CS11313 - **Fall** 2024

# Design & Analysis of Algorithms

## Voting Systems

Ibrahim Albluwi

# Problem Overview

**Ranked-Choice Voting.** Given $n$ candidates and $m$ lists that rank the candidates according to the preferences of $m$ voters. The goal is to aggregate the rankings (e.g. to pick a winner).

**Why Should Computer Scientists Care?**

- Writing software for **voting systems.**

  **Examples.** Elections, book or movie rankings, online competitions, ranking of feature requests, crowdsourcing (e.g. multiple experts ranking the quality of articles), etc.

- Writing algorithms that **aggregate** the results of **multiple algorithms.**

  **Examples.** Ensemble methods in machine learning, a search engine aggregating the ranking of search results based on multiple heuristics, etc.

- Interesting theoretical **computational results.**

  **Examples.** What is the computational complexity for manipulating a certain voting system? What is the most efficient way to find the winner in a certain voting system?

# Many Possible Systems!

**Dictatorship.** The vote of only a certain voter counts. Everything else is neglected!

**Example:**
only voter **13**
gets to decide!

| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | **13** | 14 |

# Many Possible Systems!

**Dictatorship.** The vote of a certain voter counts. Everything else is neglected!

**Plurality.** The candidate ranked first most is the winner.
Note that the winner might not be a *majority* winner (in the example below, the winner is ranked first by only 5/14 voters)
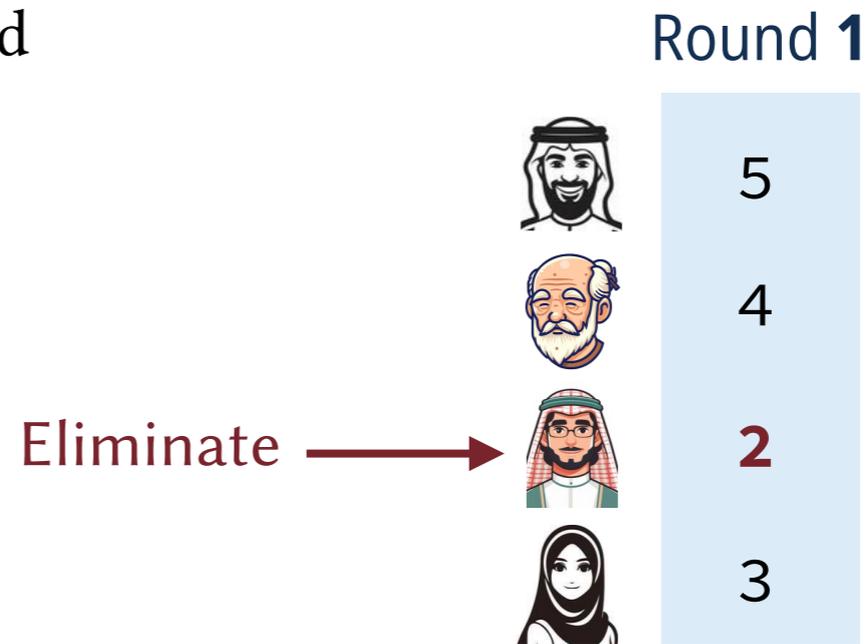
winner!

| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 |
|---|---|---|---|---|---|---|---|---|----|----|----|----|----|

# Many Possible Systems!

**Dictatorship.** The vote of a certain voter counts. Everything else is neglected!

**Plurality.** The candidate ranked first most is the winner.

**Instant-Runoff.** While there is no majority winner, eliminate the lowest candidate and transfer his/her votes.
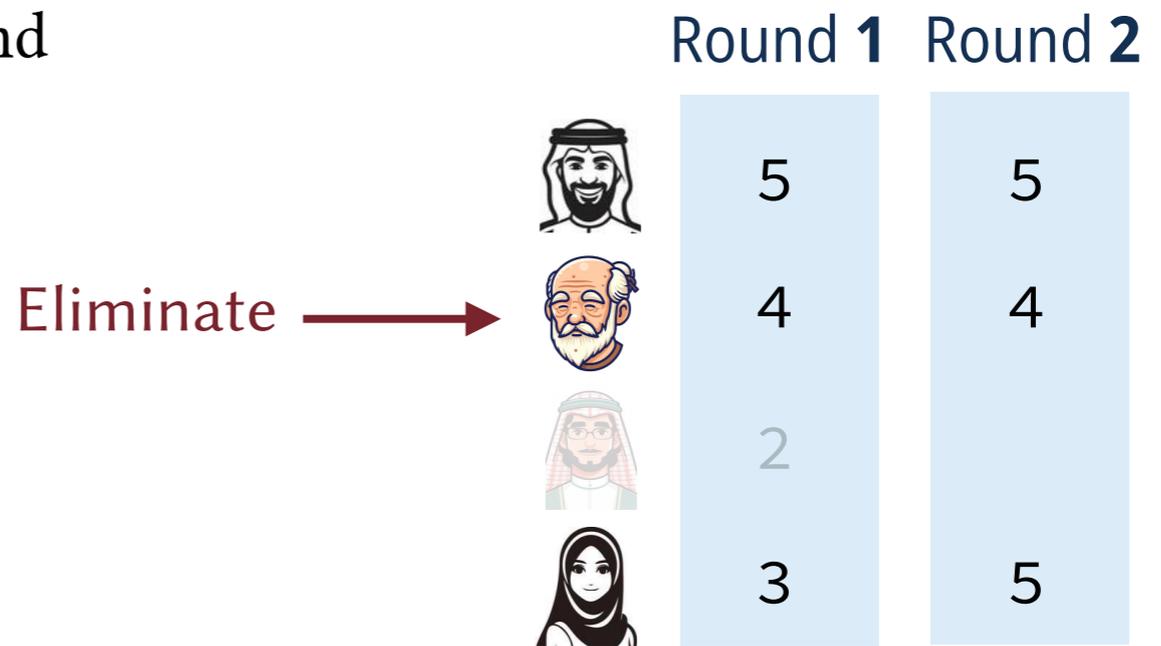
Round **1**

5

4

Eliminate ⟶ **2**

3

| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|

# Many Possible Systems!

**Dictatorship.** The vote of a certain voter counts. Everything else is neglected!

**Plurality.** The candidate ranked first most is the winner.

**Instant-Runoff.** While there is no majority winner, eliminate the lowest candidate and transfer his/her votes.
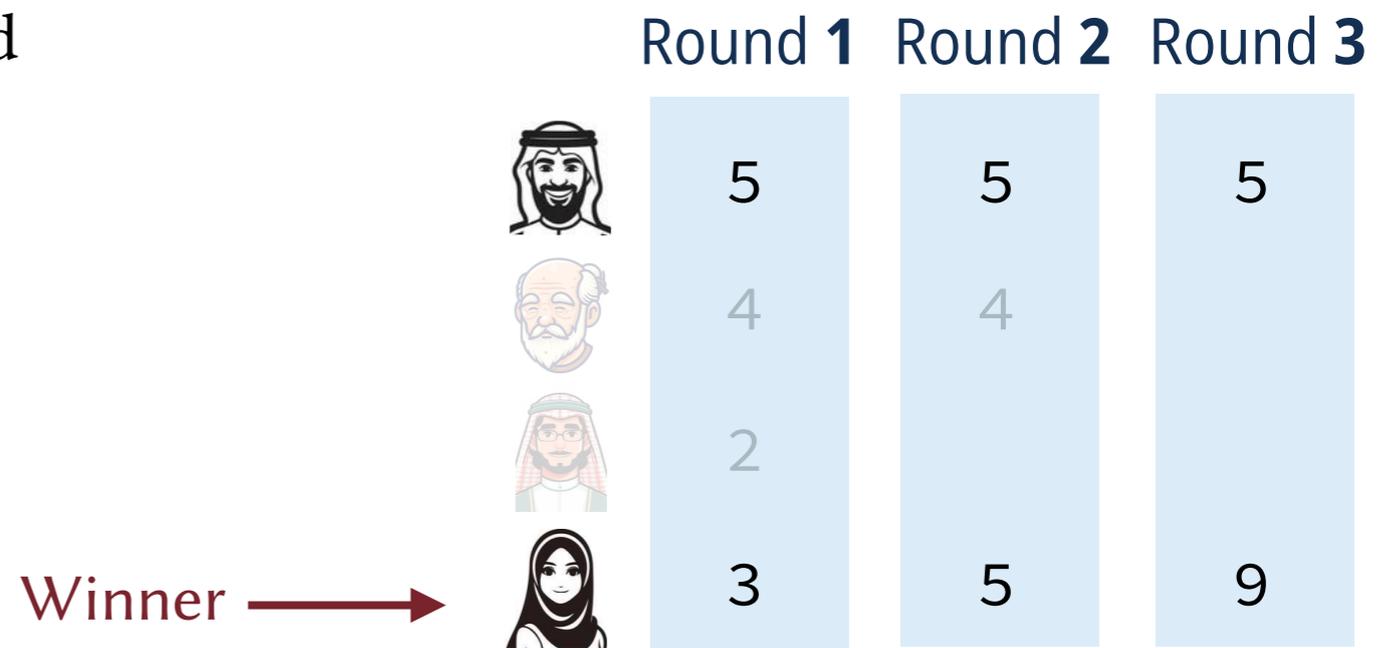
Round **1**  Round **2**

| | Round 1 | Round 2 |
|---|---|---|
| | 5 | 5 |
| Eliminate → | 4 | 4 |
| | 2 | |
| | 3 | 5 |

| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|

# Many Possible Systems!

**Dictatorship.** The vote of a certain voter counts. Everything else is neglected!

**Plurality.** The candidate ranked first most is the winner.

**Instant-Runoff.** While there is no majority winner, eliminate the lowest candidate and transfer his/her votes.

|  | Round **1** | Round **2** | Round **3** |
|---|---|---|---|
|  | 5 | 5 | 5 |
|  | 4 | 4 |  |
|  | 2 |  |  |
| Winner → | 3 | 5 | 9 |

| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|

# Many Possible Systems!

**Borda Count.** Assign $n-1$ points each time ranked 1st, $n-2$ when ranked 2nd, etc.

Borda count: 18

**Borda Count.** (5x3) + (3x1) + (6x0) = **18**

| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 |
|---|---|---|---|---|---|---|---|---|----|----|----|----|----|

# Many Possible Systems!

**Borda Count.** Assign $n - 1$ points each time ranked 1st, $n - 2$ when ranked 2nd, etc.

| | | | | |
|---|---|---|---|---|
| Borda count | 18 | | 23 | |

**Borda Count.** (3x3) + (6x2) + (2x1) + (3x0) = **23**

# Many Possible Systems!

**Dictatorship.** The vote of a certain voter counts. Everything else is neglected!

**Plurality.** The candidate ranked first most is the winner.

**Instant-Runoff.** While there is no majority winner, eliminate the lowest candidate and transfer his/her votes.

**Borda Count.** Assign $n-1$ points each time ranked 1st,
$n-2$ when ranked 2nd, etc.

| | Borda count | 18 | 24 | 23 | |
|---|---|---|---|---|---|

**Borda Count.** (4x3) + (5x2) + (2x1) + (3x0) = **24**

# Many Possible Systems!

**Borda Count.** Assign $n - 1$ points each time ranked 1st, $n - 2$ when ranked 2nd, etc.

| | | | |
|---|---|---|---|
| Borda count | 18 | 24 | 23 | 19 |

**Borda Count.** (2x3) + (3x2) + (7x1) + (2x0) = **19**

| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 |
|---|---|---|---|---|---|---|---|---|----|----|----|----|----|

# Many Possible Systems!

**Dictatorship.** The vote of a certain voter counts. Everything else is neglected!

**Plurality.** The candidate ranked first most is the winner.

**Instant-Runoff.** While there is no majority winner, eliminate the lowest candidate and transfer his/her votes.

**Borda Count.** Assign $n-1$ points each time ranked 1st, $n-2$ when ranked 2nd, etc.

| Borda count | 18 | **24** | 23 | 19 |

winner!

| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 |
|---|---|---|---|---|---|---|---|---|----|----|----|----|----|

# Many Possible Systems!

**Dictatorship.** The vote of a certain voter counts. Everything else is neglected!

**Plurality.** The candidate ranked first most is the winner.

**Instant-Runoff.** While there is no majority winner, eliminate the lowest candidate and transfer his/her votes.

**Borda Count.** Assign $n-1$ points each time ranked 1st, $n-2$ when ranked 2nd, etc.

**Sequential Pairwise Voting.**
Head-to-head challenges between pairs of candidates (who is ranked higher more?) Losers are eliminated.
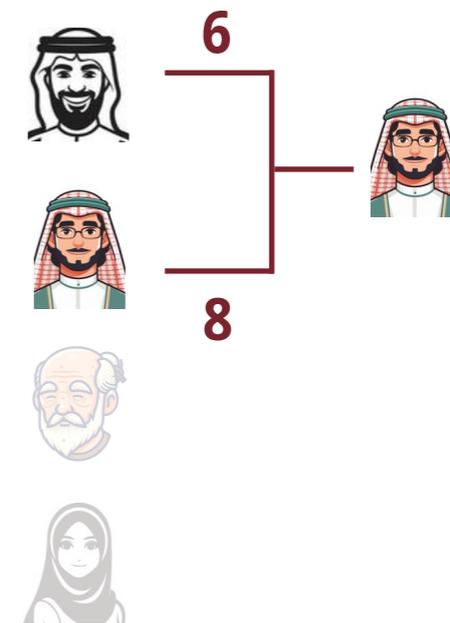
# Many Possible Systems!

**Dictatorship.** The vote of a certain voter counts. Everything else is neglected!

**Plurality.** The candidate ranked first most is the winner.

**Instant-Runoff.** While there is no majority winner, eliminate the lowest candidate and transfer his/her votes.

**Borda Count.** Assign $n - 1$ points each time ranked 1st, $n - 2$ when ranked 2nd, etc.

## Sequential Pairwise Voting.

Head-to-head challenges between pairs of candidates (who is ranked higher more?) Losers are eliminated.



| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 |
|---|---|---|---|---|---|---|---|---|----|----|----|----|----|

# Many Possible Systems!

**Dictatorship.** The vote of a certain voter counts. Everything else is neglected!
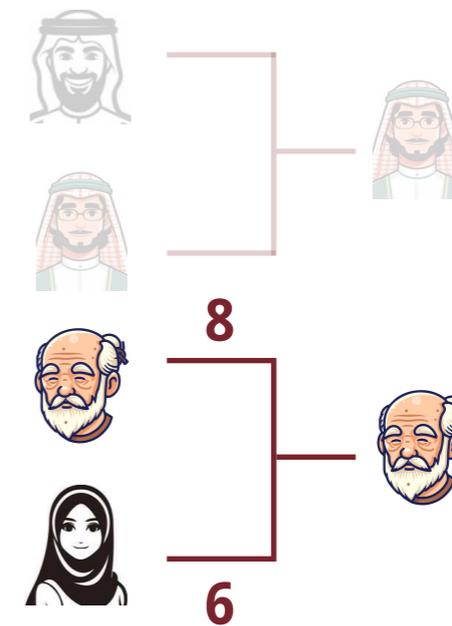
**Plurality.** The candidate ranked first most is the winner.

**Instant-Runoff.** While there is no majority winner, eliminate the lowest candidate and transfer his/her votes.

**Borda Count.** Assign $n-1$ points each time ranked 1st, $n-2$ when ranked 2nd, etc.

## Sequential Pairwise Voting.

Head-to-head challenges between pairs of candidates (who is ranked higher more?) Losers are eliminated.



| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|

# Many Possible Systems!

**Dictatorship.** The vote of a certain voter counts. Everything else is neglected!
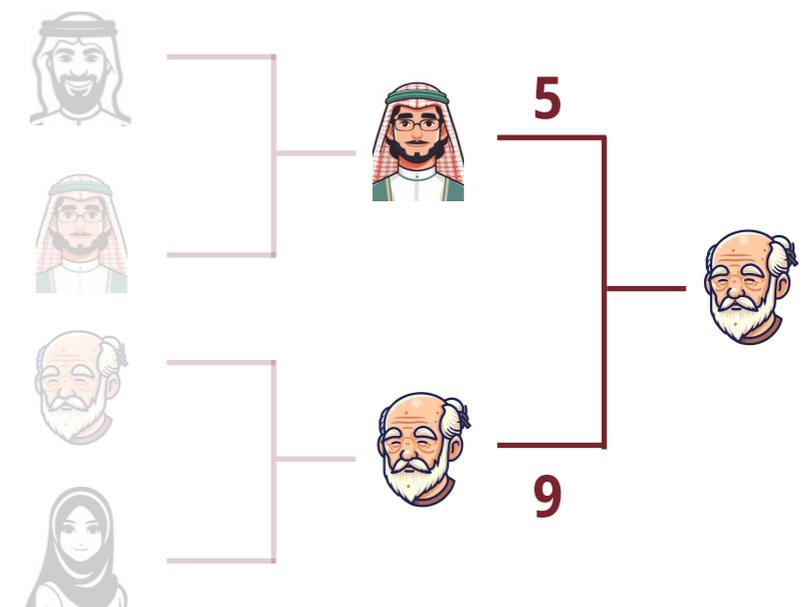
**Plurality.** The candidate ranked first most is the winner.

**Instant-Runoff.** While there is no majority winner, eliminate the lowest candidate and transfer his/her votes.

**Borda Count.** Assign $n-1$ points each time ranked 1st, $n-2$ when ranked 2nd, etc.

## Sequential Pairwise Voting.

Head-to-head challenges between pairs of candidates (who is ranked higher more?) Losers are eliminated.

A second attempt!

8

6

| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 |
|---|---|---|---|---|---|---|---|---|----|----|----|----|----|

# Many Possible Systems!

**Dictatorship.** The vote of a certain voter counts. Everything else is neglected!

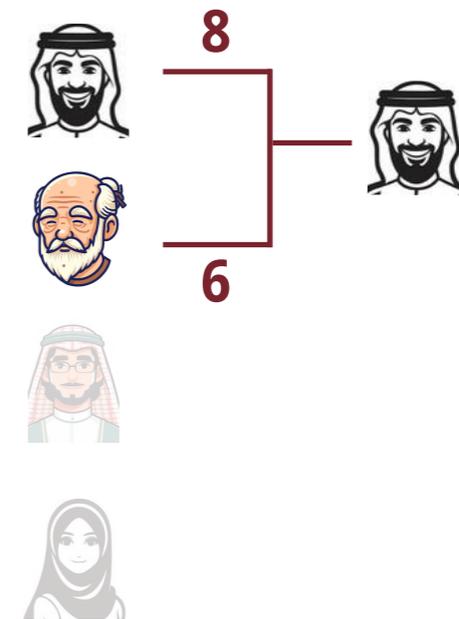**Plurality.** The candidate ranked first most is the winner.

**Instant-Runoff.** While there is no majority winner, eliminate the lowest candidate and transfer his/her votes.

**Borda Count.** Assign $n - 1$ points each time ranked 1st, $n - 2$ when ranked 2nd, etc.

**Sequential Pairwise Voting.**
Head-to-head challenges between pairs of candidates (who is ranked higher more?) Losers are eliminated.

A second attempt!

6

8



| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 |
|---|---|---|---|---|---|---|---|---|----|----|----|----|----|

# Many Possible Systems!

**Dictatorship.** The vote of a certain voter counts. Everything else is neglected!

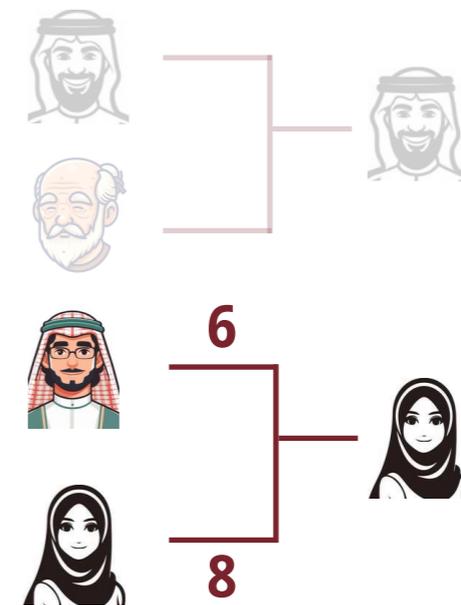**Plurality.** The candidate ranked first most is the winner.

**Instant-Runoff.** While there is no majority winner, eliminate the lowest candidate and transfer his/her votes.

**Borda Count.** Assign $n-1$ points each time ranked 1st, $n-2$ when ranked 2nd, etc.

**Sequential Pairwise Voting.**
Head-to-head challenges between pairs of candidates (who is ranked higher more?) Losers are eliminated.

A second attempt!

5

9



| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 |
|---|---|---|---|---|---|---|---|---|----|----|----|----|----|

# Many Possible Systems!

**Dictatorship.** The vote of a certain voter counts. Everything else is neglected!

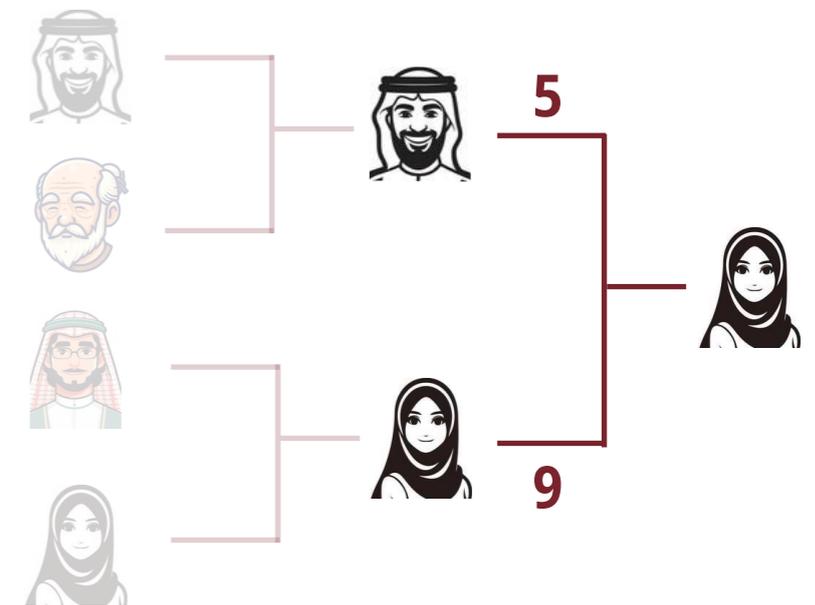**Plurality.** The candidate ranked first most is the winner.

**Instant-Runoff.** While there is no majority winner, eliminate the lowest candidate and transfer his/her votes.

**Borda Count.** Assign $n-1$ points each time ranked 1st, $n-2$ when ranked 2nd, etc.

## Sequential Pairwise Voting.
Head-to-head challenges between pairs of candidates (who is ranked higher more?) Losers are eliminated.

Winner depends on choice of pairs!

5

9

| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 |
|---|---|---|---|---|---|---|---|---|----|----|----|----|----|

# Many Possible Systems!

**Dictatorship.** The vote of a certain voter counts. Everything else is neglected!
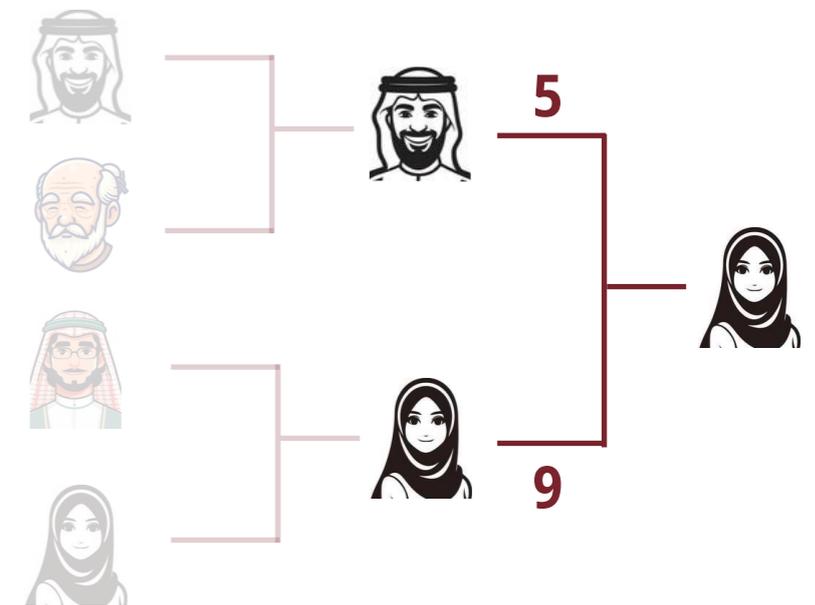
**Plurality.** The candidate ranked first most is the winner.

**Instant-Runoff.** While there is no majority winner, eliminate the lowest candidate and transfer his/her votes.

**Borda Count.** Assign $n - 1$ points each time ranked 1st, $n - 2$ when ranked 2nd, etc.

**Sequential Pairwise Voting.** Head-to-head challenges between pairs of candidates (who is ranked higher more?) Losers are eliminated.

**Etc.**

🤔
## Which voting system is the best ?

# Let's Vote!

Rank each of the following voting systems based on which one you prefer more.

- Dictatorship.
- Plurality.
- Instant-Runoff.
- Borda Count.
- Sequential Pairwise Voting.

😫 **Dilemma.** Which voting system should we use to elect the best voting system?

# Desirable Properties

**Anonymity.** The voting system returns the same outcome if the voters are relabeled. I.e. it does not matter who the voters are.

**Non-Examples.** If certain voters have *veto* rights, the electoral college (state votes have different *weight* based on geography and population.)

# Desirable Properties

**Anonymity.** The voting system returns the same outcome if the voters are relabeled.
I.e. it does not matter who the voters are.

**Neutrality.** If the labels assigned to each outcome are permuted arbitrarily, the returned result is permuted in the same way.
I.e. it does not matter who the candidates are.

**Non-Example.** In case of a tie, candidate $X$ is selected (e.g. in a parliament: if no new policy gets a majority, the currently used policy is selected).

# Desirable Properties

**Anonymity.** The voting system returns the same outcome if the voters are relabeled.
I.e. it does not matter who the voters are.

**Neutrality.** If the labels assigned to each outcome are permuted arbitrarily, the returned result is permuted in the same way.
I.e. it does not matter who the candidates are.

**Majority Rule.** The candidate ranked $1^{st}$ by more than $50\%$ of voters must win.

Which of the following methods satisfies the *majority* rule? Mark all that applies.

**A.** Dictatorship.

**B.** Plurality.

**C.** Instant-Runoff.

**D.** Borda Count.

**E.** Sequential Pairwise Voting.

# Exercise

Which of the following methods satisfies the *majority* rule? Mark all that applies.

**X** Dictatorship.

**✔** Plurality.

**✔** Instant-Runoff. — majority winner wins immediately.

**D.** Borda Count.

**E.** Sequential Pairwise Voting.

# Exercise

Which of the following methods satisfies the *majority* rule? Mark all that applies.

✗    Dictatorship.

✓    Plurality.

✓    Instant-Runoff.

✗    Borda Count.

**E.**   Sequential Pairwise Voting.

**9 Votes**    **5 Votes**

Majority Winner

Borda Count Winner

$\text{Borda count} = (5 \times 3) + (9 \times 2) = 33$

$\text{Borda count} = (9 \times 3) + (5 \times 0) = 27$

Which of the following methods satisfies the *majority* rule? Mark all that applies.

- ✗ Dictatorship.
- ✓ Plurality.
- ✓ Instant-Runoff.
- ✗ Borda Count.
- ✓ Sequential Pairwise Voting. ⟶ majority winner wins in every head-to-head challenge.

# Desirable Properties

**Anonymity.** The voting system returns the same outcome if the voters are relabeled.
I.e. it does not matter who the voters are.

**Neutrality.** If the labels assigned to each outcome are permuted arbitrarily, the returned result is permuted in the same way.
I.e. it does not matter who the candidates are.

**Majority Rule.** The candidate ranked $1^{st}$ by more than $50\,\%$ of voters must win.

**Participation Rule.** A candidate should not lose an election as a result of having too many voters (or should not win as a result of having too few voters).

> **No-Show Paradox.** Candidate $A$ loses the election to candidate $B$ as a result of adding more voters that rank $A$ higher than $B$.

Which of the following methods satisfies the *participation* rule? Mark all that applies.

**A.** Dictatorship.

**B.** Plurality.

**C.** Instant-Runoff.

**D.** Borda Count.

**E.** Sequential Pairwise Voting.

# Exercise

Which of the following methods satisfies the *participation* rule? Mark all that applies.

✔ Dictatorship. ⟶ If you are the dictator, showing up makes your candidate win, not showing up makes them lose.

If you are not the dictator, it does not matter if you show up or not.

**B.** Plurality.

**C.** Instant-Runoff.

**D.** Borda Count.

**E.** Sequential Pairwise Voting.

# Exercise

Which of the following methods satisfies the *participation* rule? Mark all that applies.

✔ Dictatorship.

✔ Plurality. ————————→ Not showing up decreases the number of times your candidate is ranked first.

C. Instant-Runoff.

✔ Borda Count. ————————→ Not showing up decreases your candidate's Borda count.

✔ Sequential Pairwise Voting. ————————→ Not showing up decreases your candidate's ability to win in head-to-head matches.

Which of the following methods satisfies the *participation* rule? Mark all that applies.

- ✓ Dictatorship.
- ✓ Plurality.
- ✗ Instant-Runoff.
- ✓ Borda Count.
- ✓ Sequential Pairwise Voting.



**+6** voters that like 👩 more than 👴

| Round **1** | | | | Round **2** | | | |
|---|---|---|---|---|---|---|---|
| **25** | **15** | **15** | **39** | **25** | **15** | **15** | **39** |

👩 wins over 👴

| Round **1** | | | | Round **2** | | | |
|---|---|---|---|---|---|---|---|
| **31** | **15** | **15** | **39** | **31** | **15** | **15** | **39** |

👴 wins over 👩

# Desirable Properties

**Anonymity.** The voting system returns the same outcome if the voters are relabeled. I.e. it does not matter who the voters are.

**Neutrality.** If the labels assigned to each outcome are permuted arbitrarily, the returned result is permuted in the same way. I.e. it does not matter who the candidates are.

**Majority Rule.** The candidate ranked $1^{st}$ by more than $50\%$ of voters must win.

**Participation Rule.** A candidate should not lose an election as a result of having too many voters (or should not win as a result of having too few voters).

> **No-Show Paradox.** Candidate $A$ loses the election to candidate $B$ as a result of adding more voters that rank $A$ higher than $B$.

**Monotonicity.** Increasing a candidate's ranking should not cause that candidate to lose. Decreasing a candidate's ranking should not cause that candidate to win.

**Note.** The Participation Rule is about the effect of participating or abstaining while Monotonicity is about changing the vote.
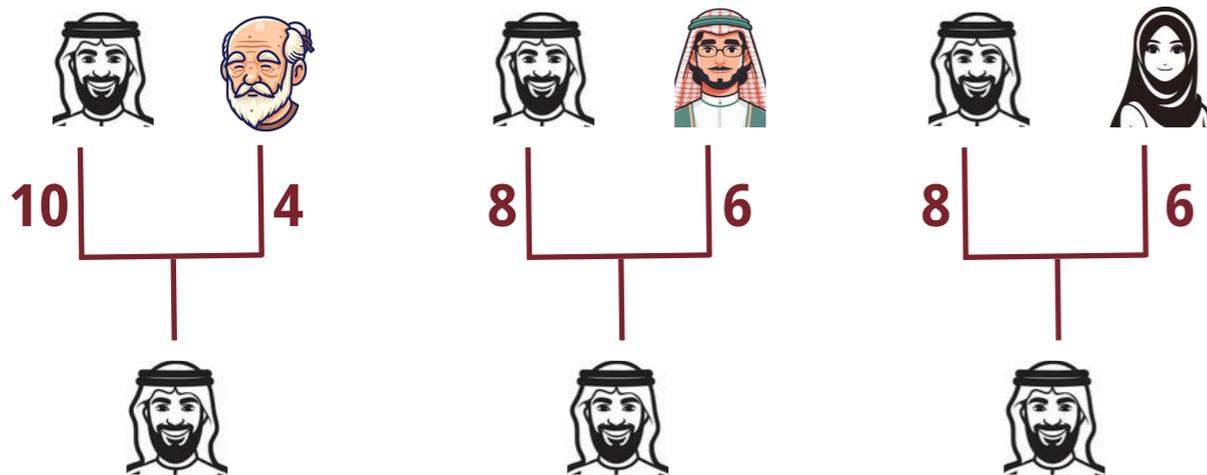
**Condorcet Winner.** The candidate who wins every head-to-head challenge against each of the other candidates.

Marquis de Condorcet
1743 – 1794

**Example.**

| 10 | 4 | | 8 | 6 | | 8 | 6 |

| 6 Votes | 2 Votes | 4 Votes | 2 Votes |
|---|---|---|---|

# Desirable Properties

**Condorcet Winner.** The candidate who wins every head-to-head challenge against each of the other candidates.

**Condorcet Method.** A method that always picks the Condorcet winner (if one exists!).

**Condorcet Paradox.** A Condorcet winner does not always exist!



Marquis de Condorcet
1743 – 1794

**Example.** Every candidate loses to one other candidate.

**4 Votes**  **4 Votes**  **4 Votes**  **4 Votes**

# Desirable Properties

**Condorcet Winner.** The candidate who wins every head-to-head challenge against each of the other candidates.
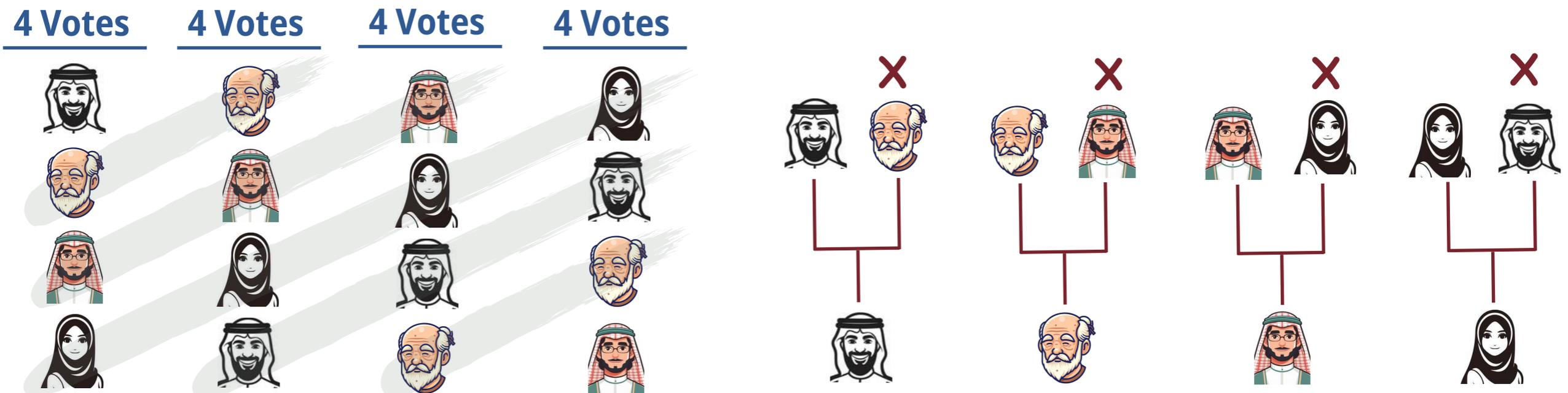
**Condorcet Method.** A method that always picks the Condorcet winner (if one exists!).

**Condorcet Paradox.** A Condorcet winner does not always exist!

**Good News.** Difficult to occur if the number of voters is large.

Marquis de Condorcet
1743 – 1794

**Example.** Every candidate loses to one other candidate.

| 4 Votes | 4 Votes | 4 Votes | 4 Votes |
| --- | --- | --- | --- |

A Condorcet Cycle

# Desirable Properties

**Condorcet Winner.** The candidate who wins <span style="color:red">every</span> head-to-head challenge against each of the other candidates.

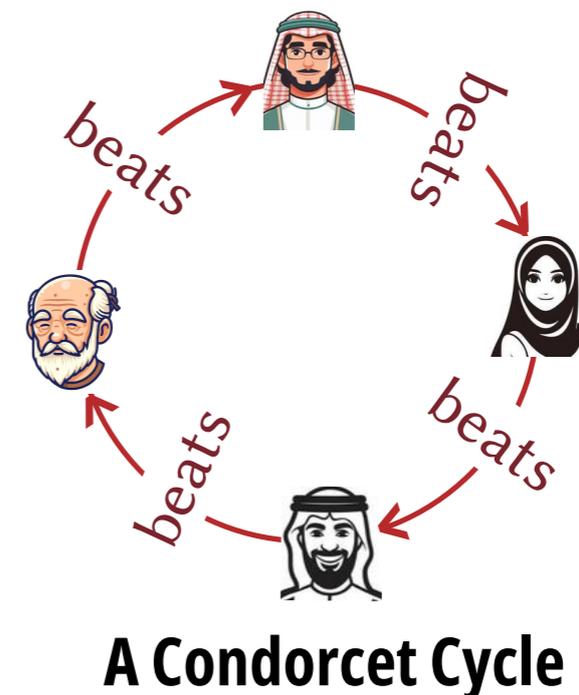**Condorcet Method.** A method that always picks the Condorcet winner (if one exists!).

**Condorcet Paradox.** A Condorcet winner does not always exist!

**Good News.** Difficult to occur if the number of voters is large.

Marquis de Condorcet
1743 – 1794

**Copeland Winner.** The candidate who wins the <span style="color:red">majority</span> head-to-head challenges against other candidates.

**Copeland Method.** A method that always picks a Copeland winner (if one exists!).

Show that all of the following methods are **not** Condorcet Methods.

X    Dictatorship.

**B.**  Plurality.

**C.**  Instant-Runoff.

**D.**  Borda Count.

# Exercise

Show that all of the following methods are **not** Condorcet Methods.

    X     Dictatorship.

    **X**     Plurality.

    C.     Instant-Runoff.

    D.     Borda Count.

**Plurality Winner**

| 10 | 4 | 8 |
|----|---|---|

Condorcet Winner

# Exercise

Show that all of the following methods are **not** Condorcet Methods.

✗   Dictatorship.

✗   Plurality.

✗   Instant-Runoff.

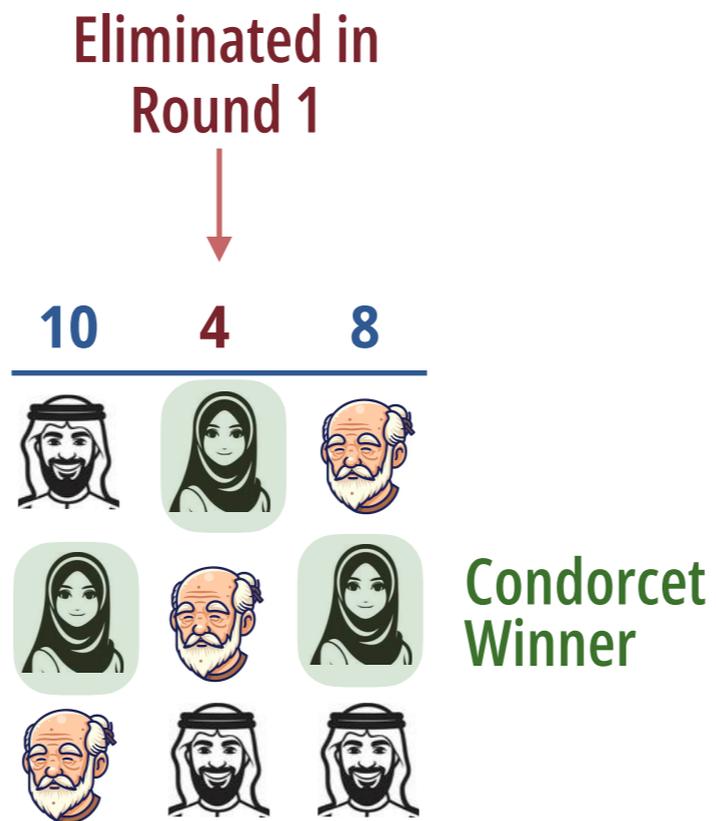D.   Borda Count.



Eliminated in Round 1

10    4    8

Condorcet Winner

# Exercise

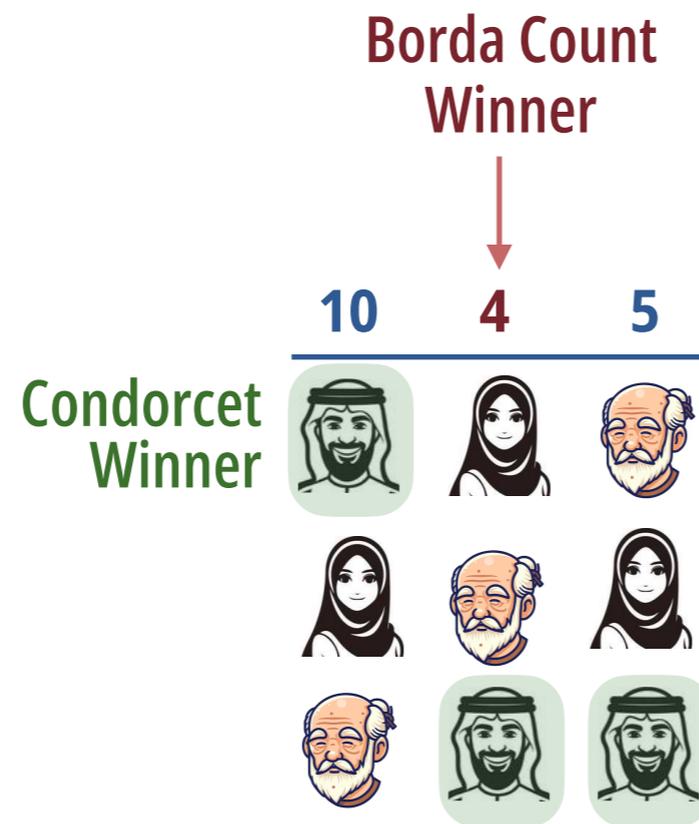Show that all of the following methods are **not** Condorcet Methods.

- ✗ Dictatorship.
- ✗ Plurality.
- ✗ Instant-Runoff.
- ✗ Borda Count.

**Independence of Irrelevant Alternatives (IIA).** When deciding on the relative ranking between $A$ and $B$, the decision should not depend on another alternative $C$.

**Implications.**

- If candidate $A$ wins an election:
  - **Removing** candidate $B$ from the ballot, must have no effect on the result.
  - **Introducing** a new candidate $B$ to the ballot, must make the winner either $A$ or $B$.
- If the voting system ranks $A$ above $B$ and voters **change $C$** to be ranked above (or below $A$), then $A$ should still be ranked above $B$.

*Morgenbesser, ordering dessert, is told by a waitress that he can choose between **blueberry** or **apple** pie. He orders **apple**. Soon the waitress comes back and explains **cherry** pie is also an option. Morgenbesser replies "In that case, I'll have **blueberry**."*

Sidney Morgenbesser

# Arrow's Impossibility Theorem

No ranked-choice voting system with $\geq 3$ candidates, can satisfy all the following three conditions together:

1. Non-dictatorship.

2. Pareto-efficiency (if $A$ is preferred to $B$ by every voter, so should the result).

3. Independence of Irrelevant Alternatives.

Kenneth Arrow (1951)

**Formal Proof.** Skipped.

**Notes.**

- No reasonable voting system should drop the non-dictatorship condition or the Pareto-efficiency condition. The practical implication of Arrow's impossibility theorem is that **reasonable voting systems do not satisfy IIA**.

- Arrow's theorem does not apply to non-ranked-choice voting systems (e.g. Approval Voting).

# Desirable Properties

**Truthfulness.** The voting system should not incentivize strategic voting.

**Strategic Voting.** Maximizing one's satisfaction with the election's results by voting against one's true preferences but in consideration of others voted.

**Gibbard-Satterthwaite Theorem.** For any voting system that is not a dictatorship and where at least three different outcomes are possible, the system sometimes incentivizes strategic voting.

**Implication.** Strategic voting situations are inevitable in reasonable voting systems.

**Note.** Arrow's theorem is a consequence of the presence of Condorcet Cycles (proof skipped), and Gibbard-Satterthwaite is a consequence of the absence IIA (i.e. of Arrow's impossibility theorem)

# Exercise

Which of the following systems is *easy* to manipulate?

- **A.** Dictatorship.
- **B.** Plurality.
- **C.** Instant-Runoff.
- **D.** Borda Count.
- **E.** Sequential Pairwise Voting.

# Hardness of Manipulation

Table 1: Complexity Results of Constructive Manipulations [2]

| Number of candidates | 2 | 3 | 4, 5, 6 | ≥ 7 |
|---|---|---|---|---|
| Borda | P | NP-c | NP-c | NP-c |
| veto | P | NP-c | NP-c | NP-c |
| STV | P | NP-c | NP-c | NP-c |
| plurality with runoff | P | NP-c | NP-c | NP-c |
| Copeland | P | P | NP-c | NP-c |
| maximin | P | P | NP-c | NP-c |
| randomized cup | P | P | P | NP-c |
| regular cup | P | P | P | P |
| plurality | P | P | P | P |

Similar to Instant Runoff →

Helping a candidate win.

Preventing a candidate from winning.

Table 2: Complexity Results of Destructive Manipulations [2]

| Number of candidates | 2 | ≥ 3 |
|---|---|---|
| STV | P | NP-c |
| plurality with runoff | P | NP-c |
| randomized cup | P | ? |
| Borda | P | P |
| veto | P | P |
| Copeland | P | P |
| maximin | P | P |
| regular cup | P | P |
| plurality | P | P |

**Source**: https://www.cs.ubc.ca/~kevinlb/teaching/cs532a%20-%202005-6/Projects/ElizabethCross.pdf