CS11921 - **Fall** 2023

Algorithm
Design & Analysis

Data Structures for Disjoint Sets

Ibrahim Albluwi

A Union-Find Data Structure

Problem. Given n items, each in a singleton set, build a data structure to support the following operations:

`UNION``(p, q)`	Merge the set containing **`p`** and the set containing **`q`** into one set.
`FIND``(p)`	Identify which set item **`p`** belongs to.
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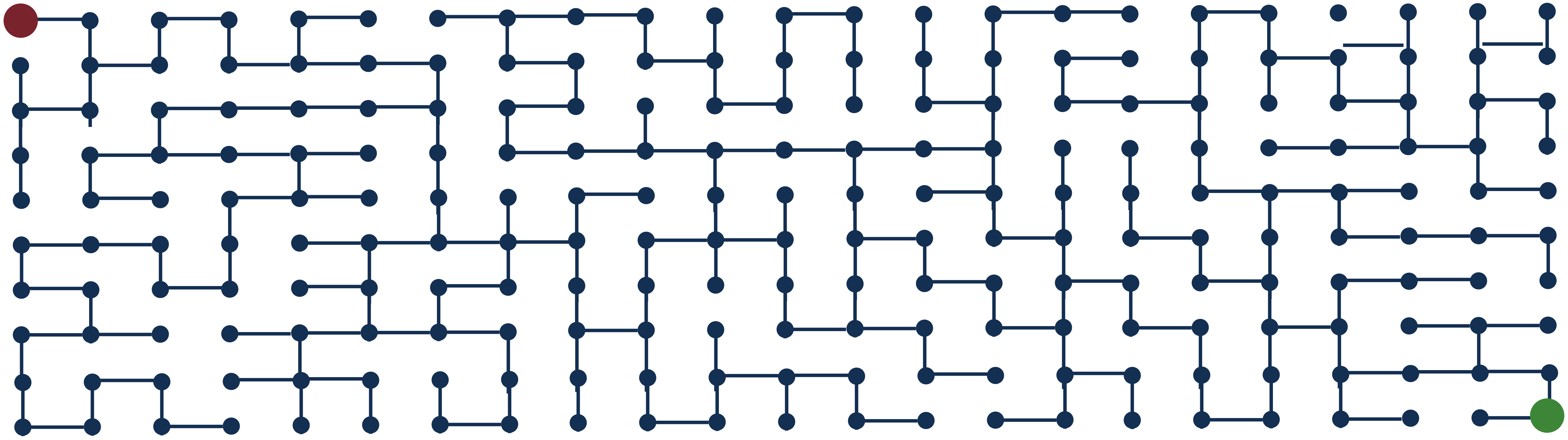

Is the **red** node connected to the **green** node?
Are all the nodes connected? Is there a cycle?

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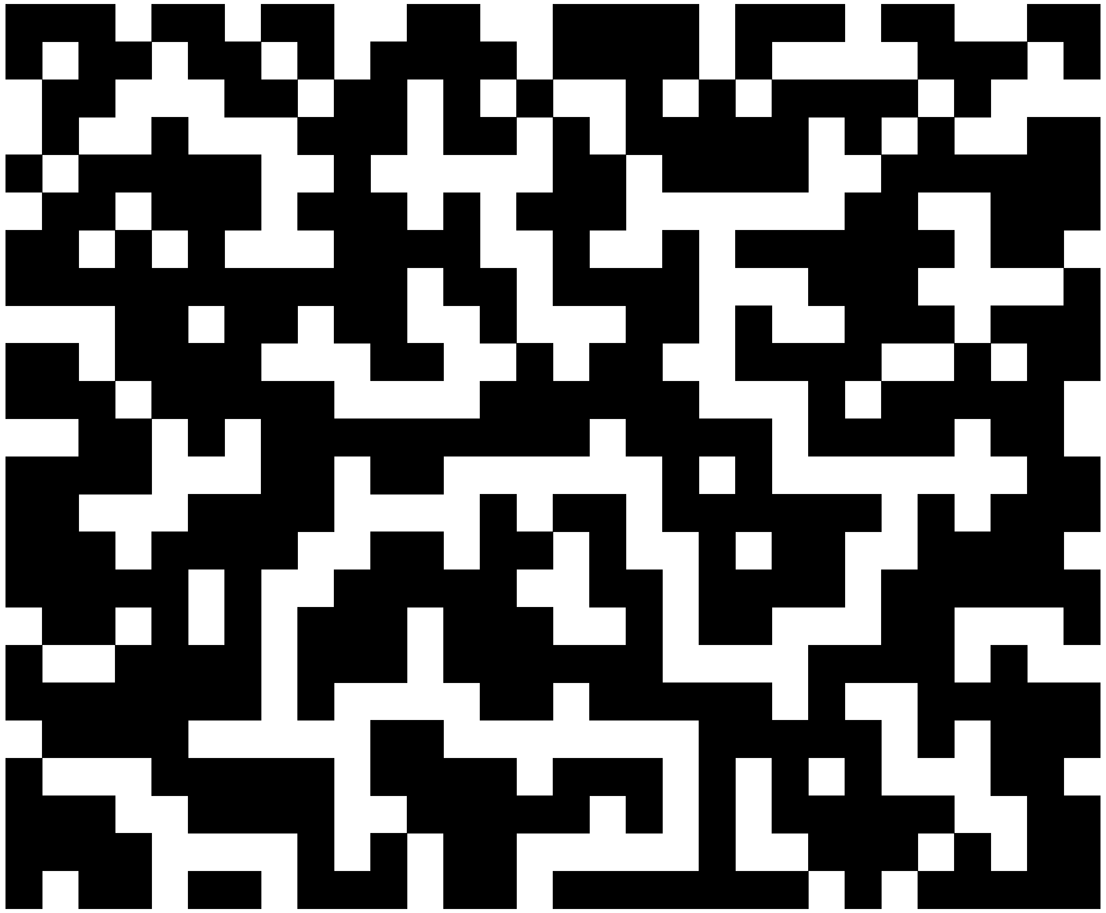

Does this plate conduct electricity?

(black = conductive material
white = insulating material)

A Simple Implementation: Array-based Quick-Find

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Example.

id	0	1	2	3	4	5	6	7	8	9
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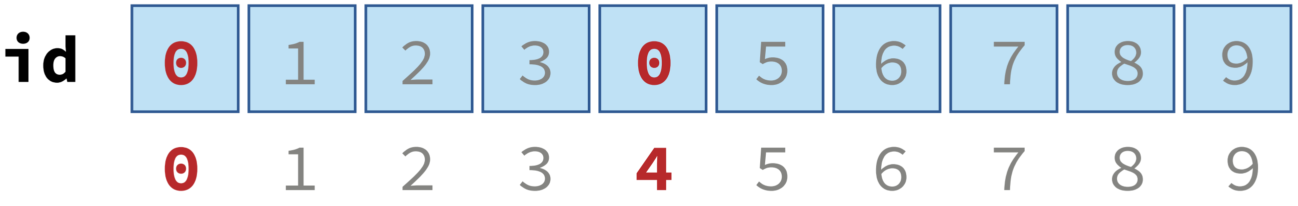
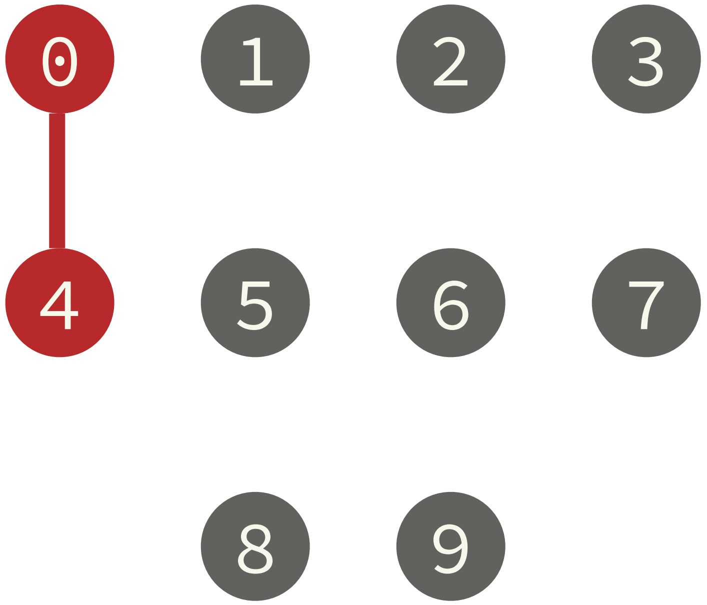


id	**0**	1	2	3	**0**	5	6	7	8	9
	0	1	2	3	**4**	5	6	7	8	9

- **`UNION`**`(0, 4)`



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id	0	1	1	3	0	5	6	7	8	9
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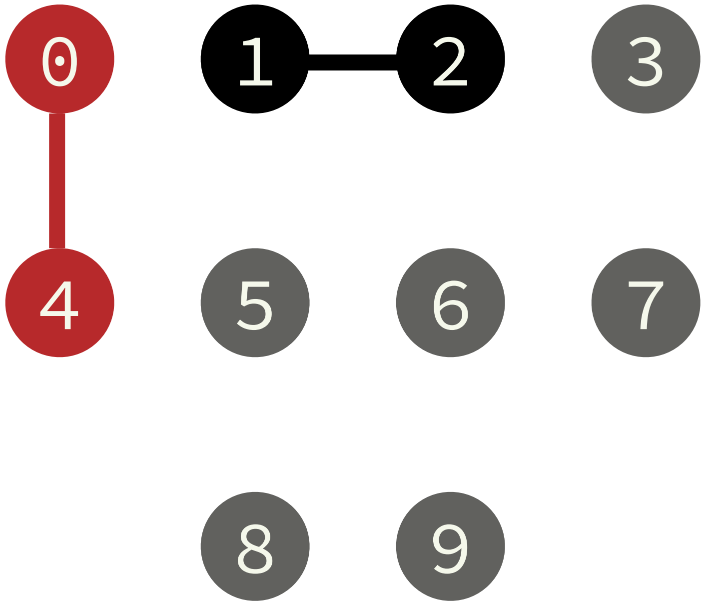
- `UNION(0, 4)`
- **`UNION`**`(1, 2)`



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Example.

id	**0**	**0**	**0**	3	**0**	5	6	7	8	9
	0	**1**	**2**	3	**4**	5	6	7	8	9

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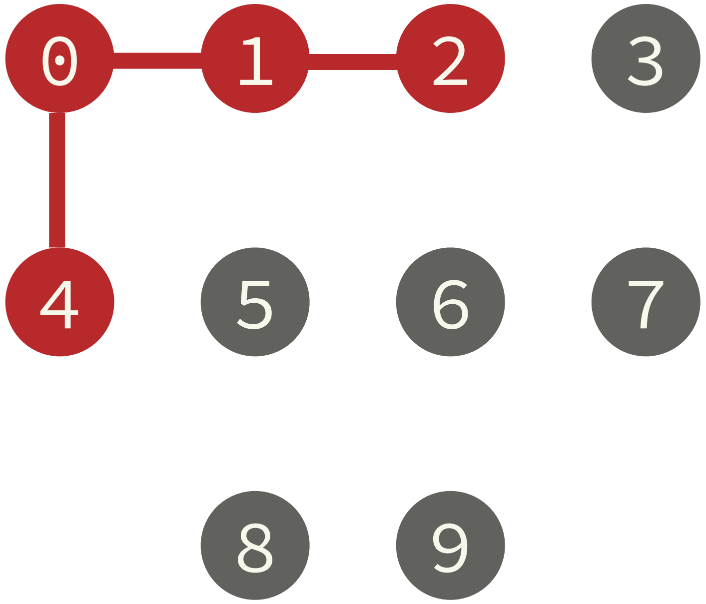

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id	0	0	0	3	0	5	6	3	8	9
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- **`UNION`**`(0, 4)`
- **`UNION`**`(1, 2)`
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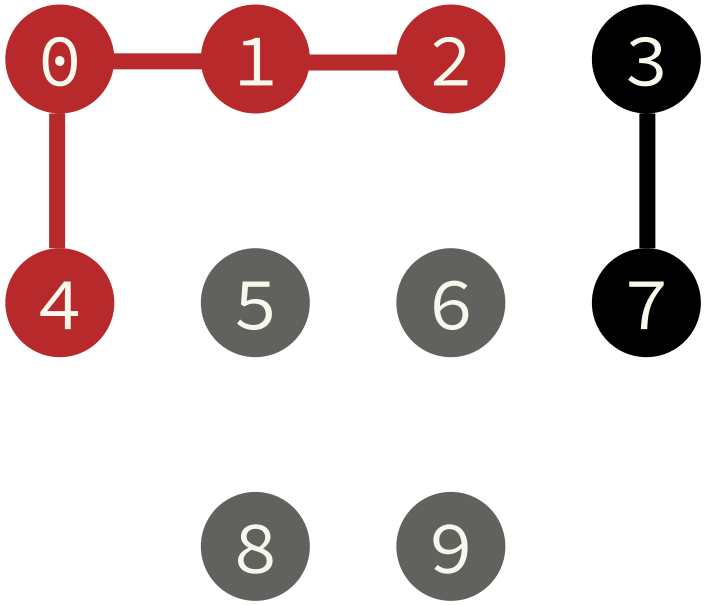
- **`UNION`**`(3, 7)`



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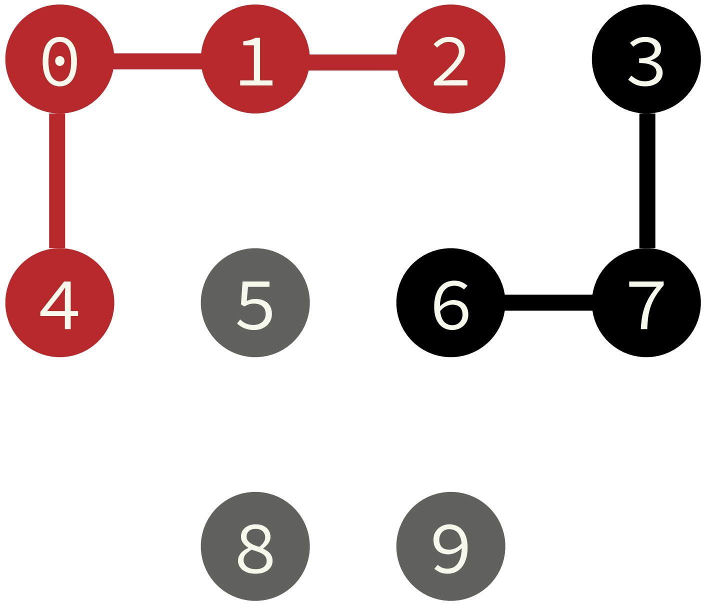
Example.

id	0	0	0	6	0	5	6	6	8	9
	0	1	2	3	4	5	6	7	8	9

- `UNION(0, 4)`
- `UNION(1, 2)`
- `UNION(0, 1)`
- `UNION(3, 7)`
- **`UNION(6, 7)`**



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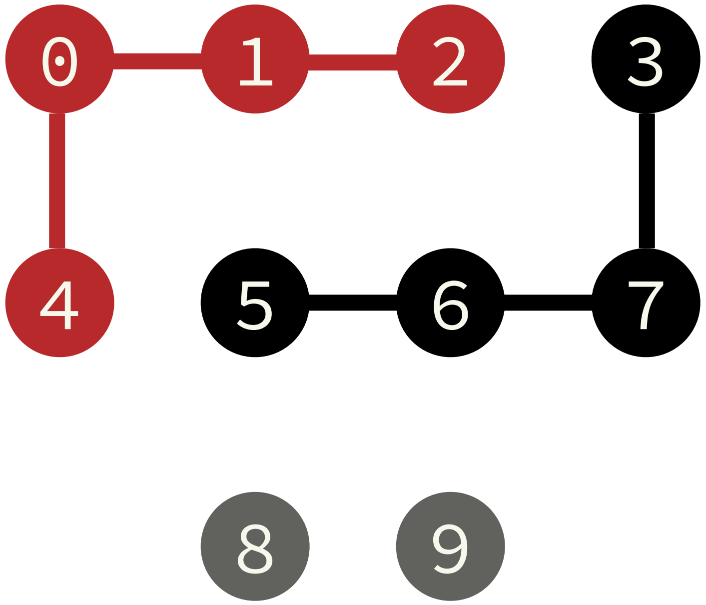
- **`UNION(5, 6)`**



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Example.

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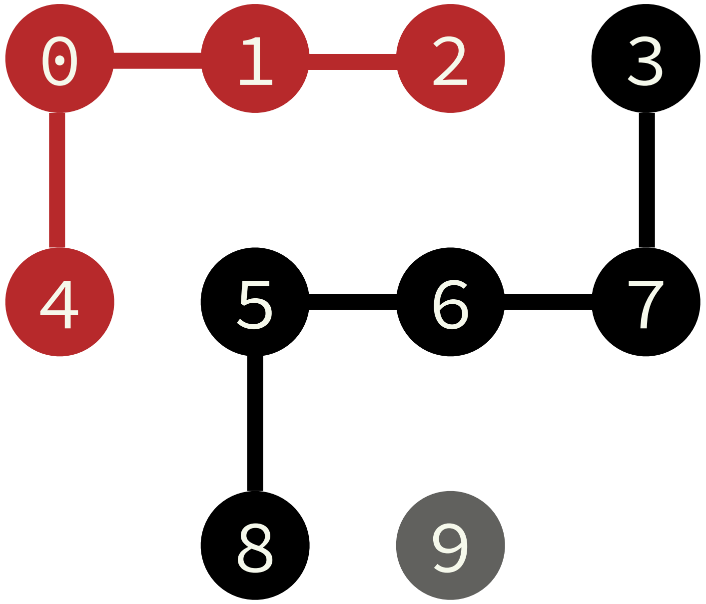

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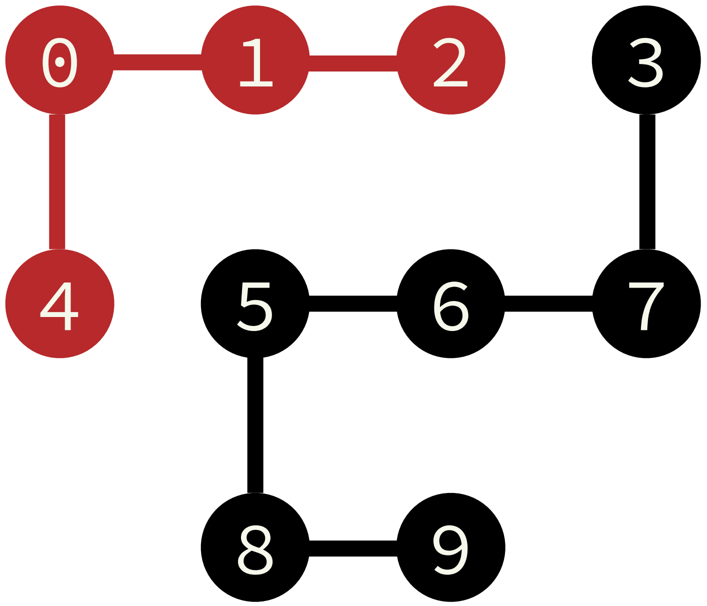

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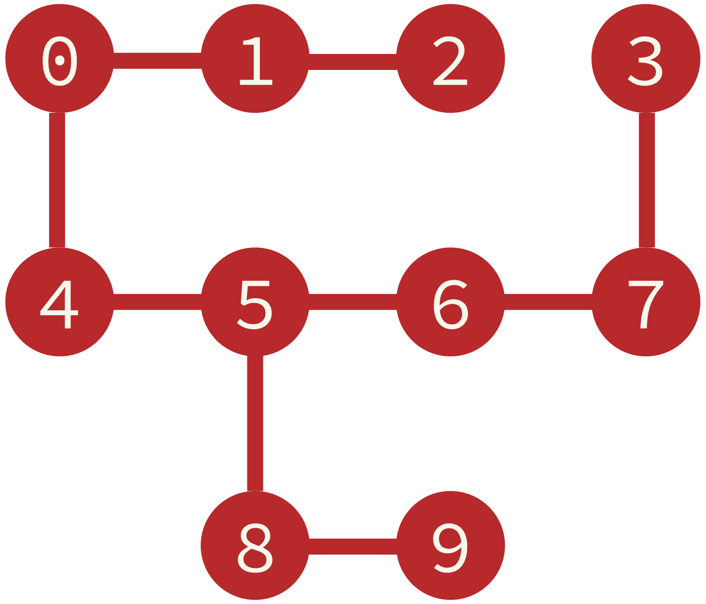

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- `UNION(4, 5)`

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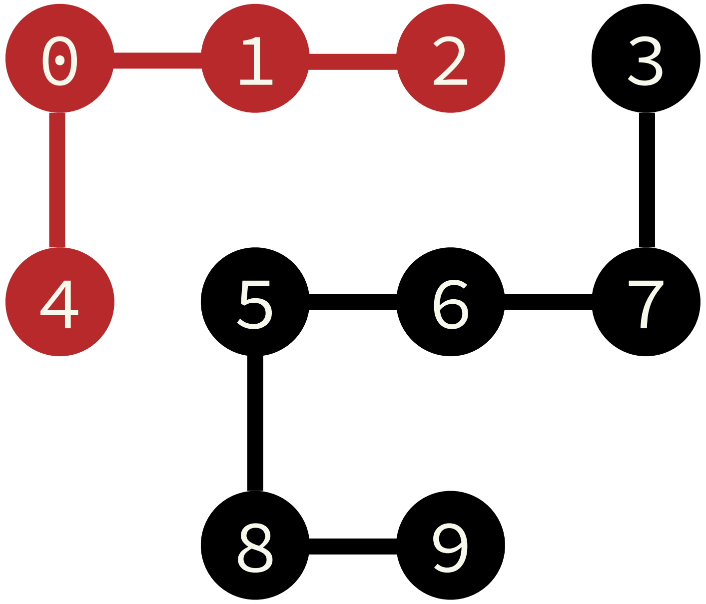

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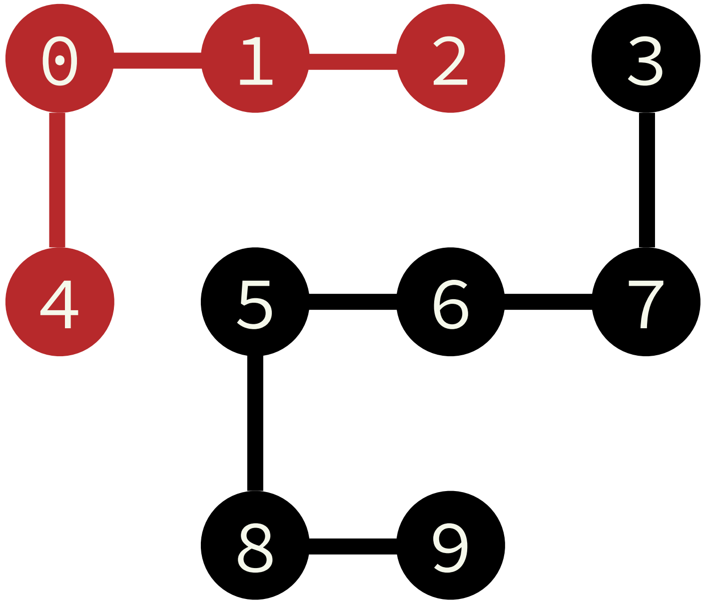

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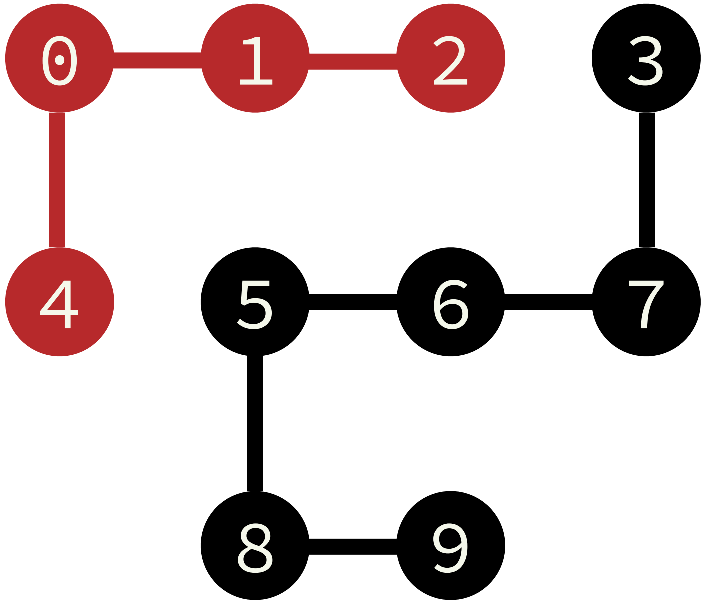

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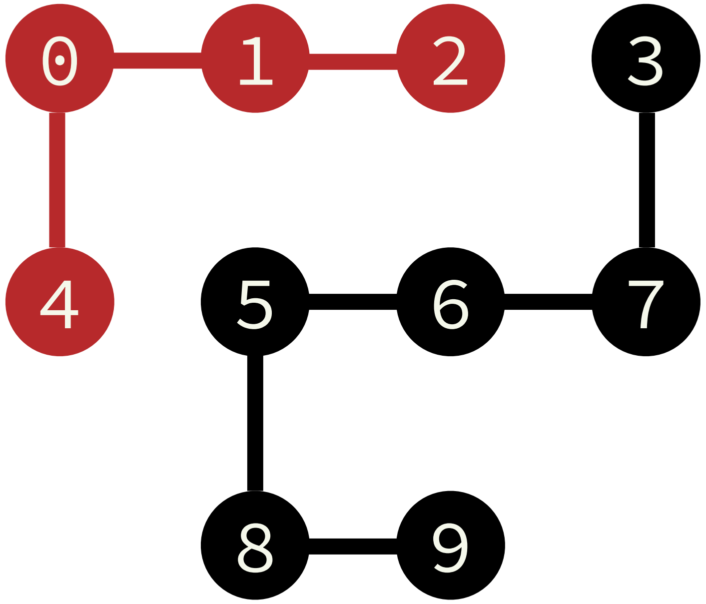

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Buggy Code?

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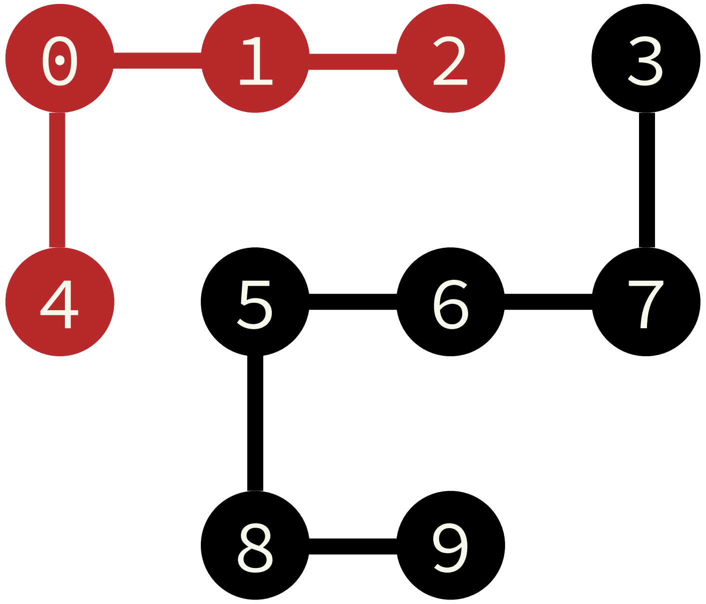

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Running Time.

`FIND:` $\Theta(1)$

`UNION:` $\Theta(n)$

Cost Model. Number of array accesses

Example.

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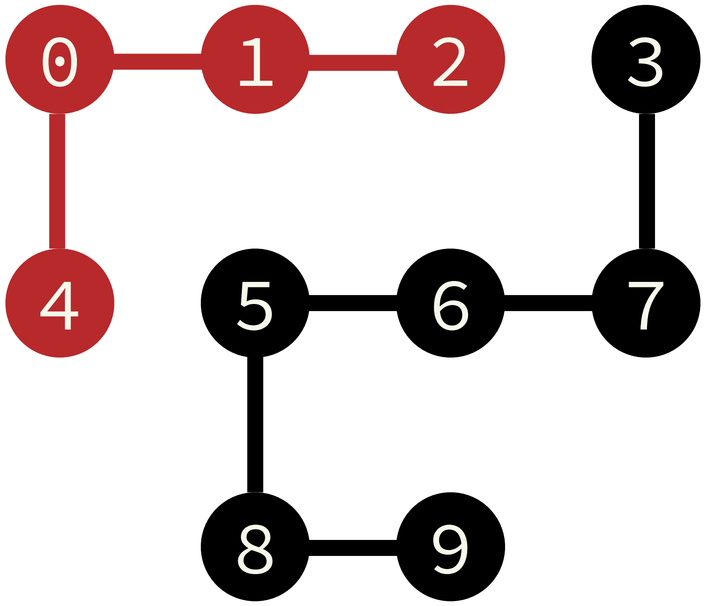

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Idea. Each set is a linked linked list.
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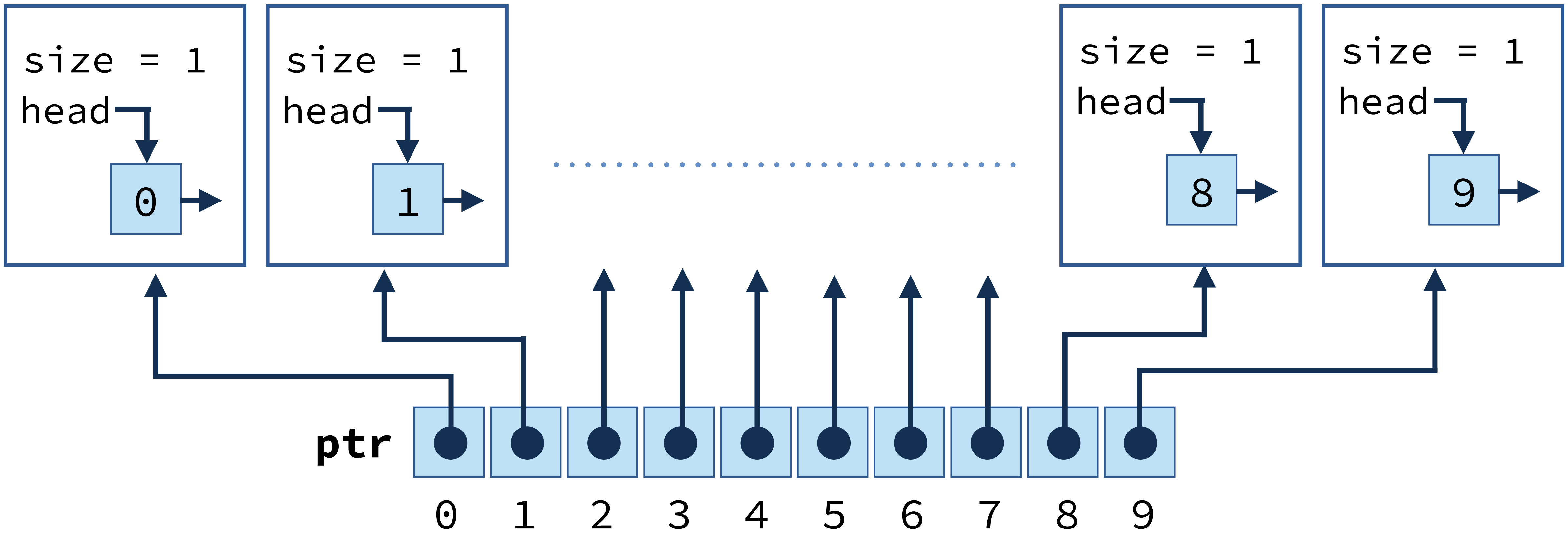

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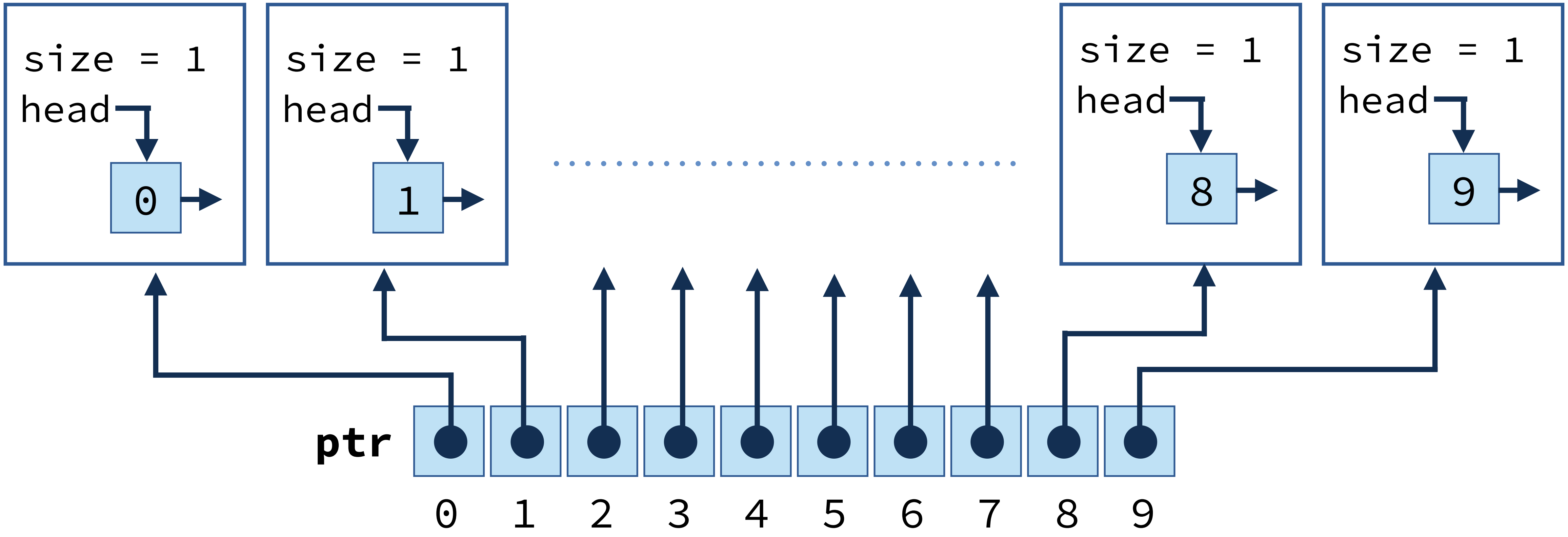

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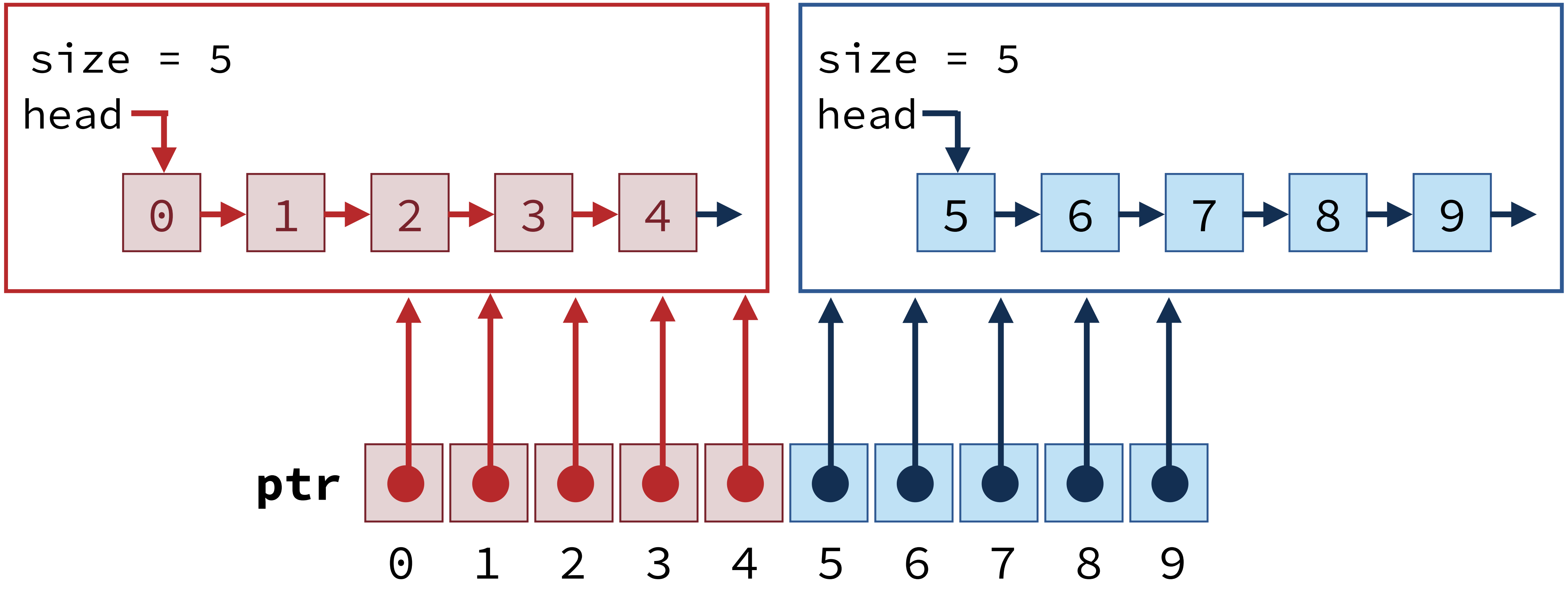
Example. Elements `0–4` are in one set and `5–9` are in another set.



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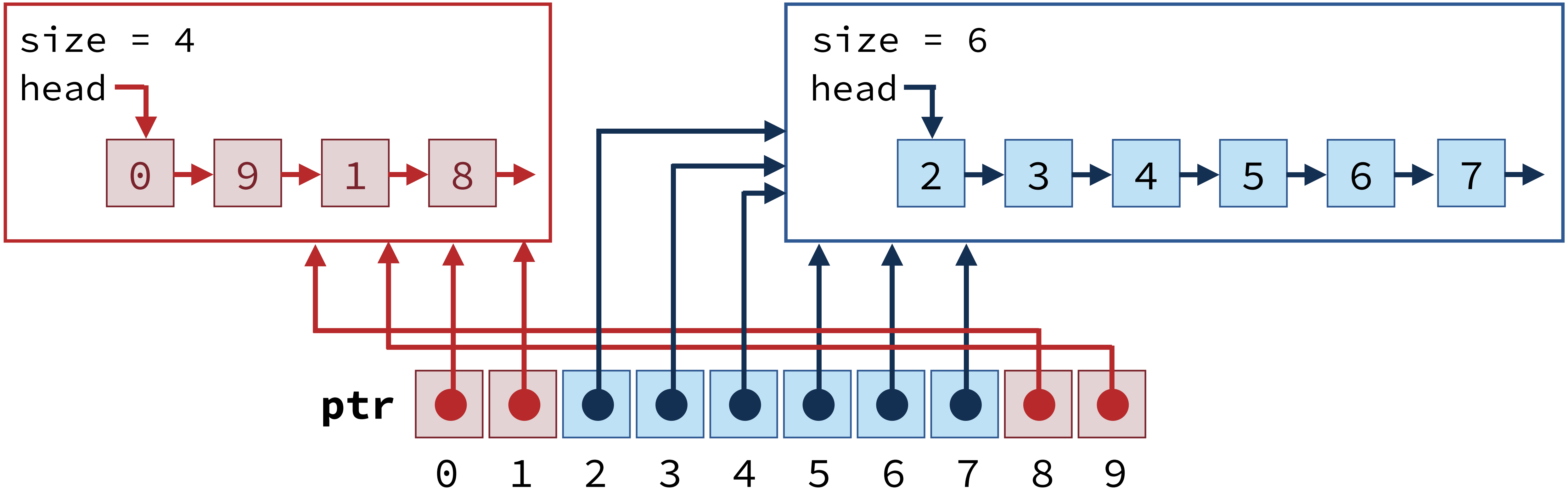
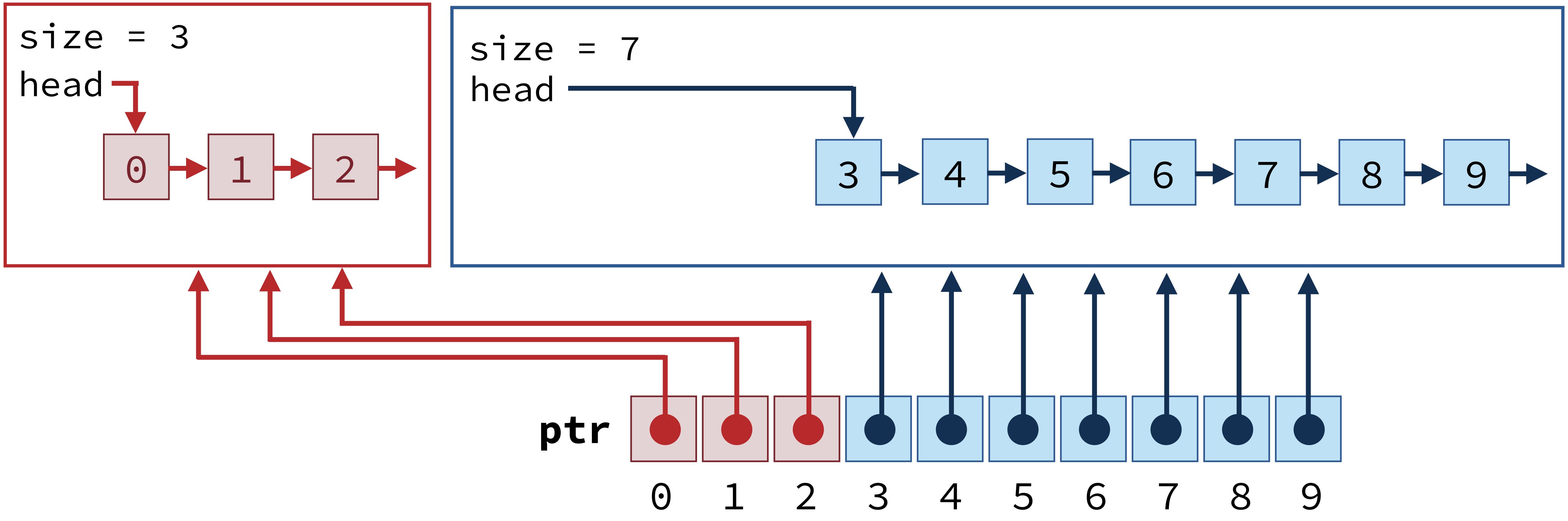
Example. Two sets: `{0,1,9,8}` and `{2,3,4,5,6,7}`.



Link-List-Based Quick-Find: **`UNION`** Example

Example. **`UNION`**`(1, 7)`



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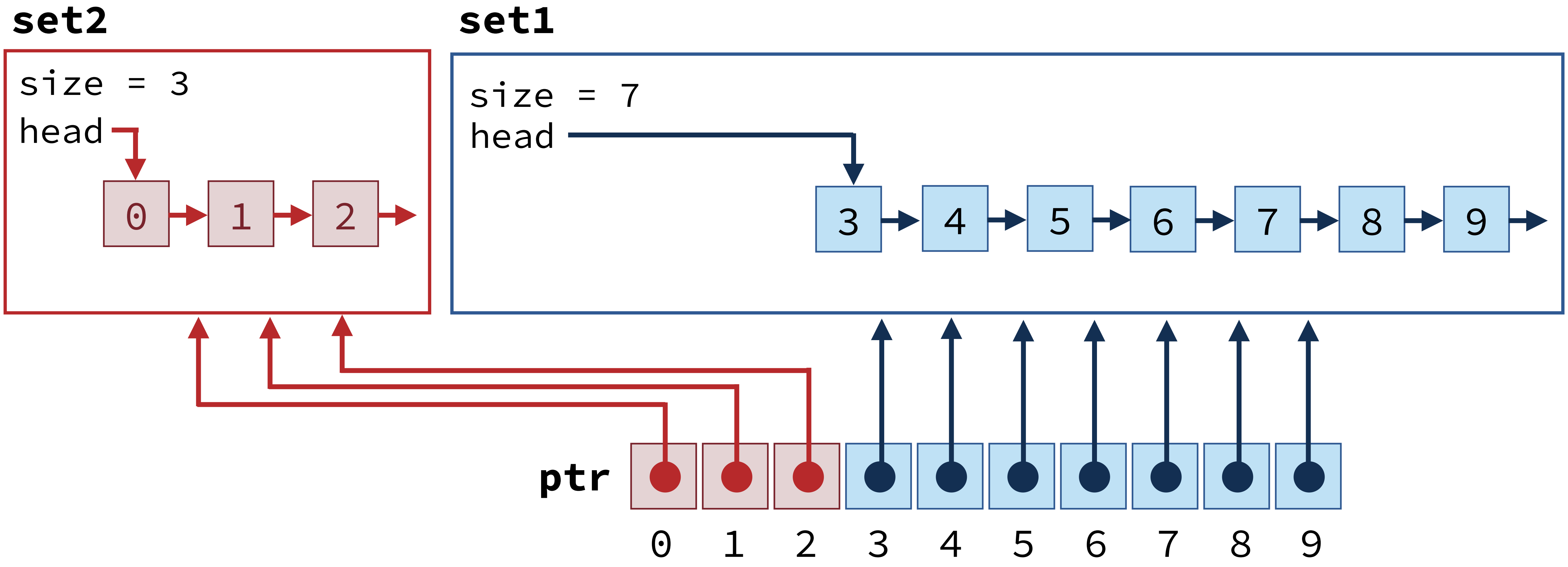

Link-List-Based Quick-Find: `UNION` Example

Example. **`UNION(1, 7)`**

1. Make `set2` point at the smaller set and `set1` at the larger set.
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   `ptr[e] = set1`
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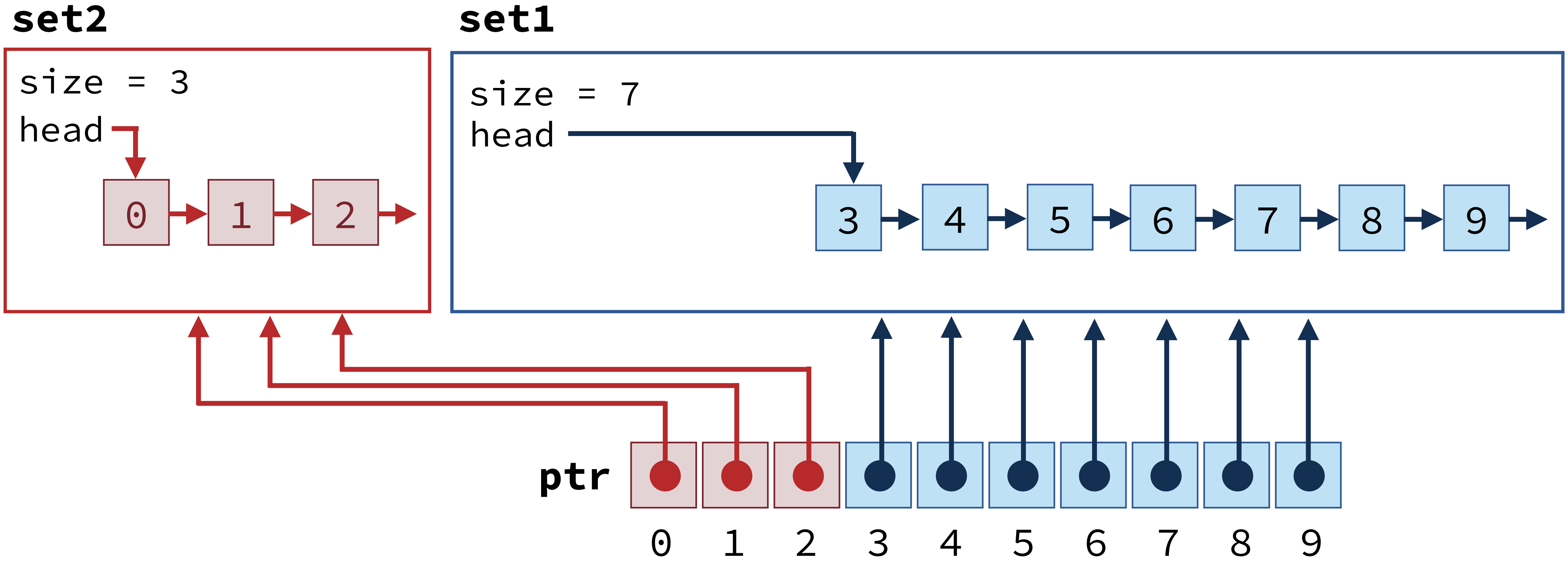

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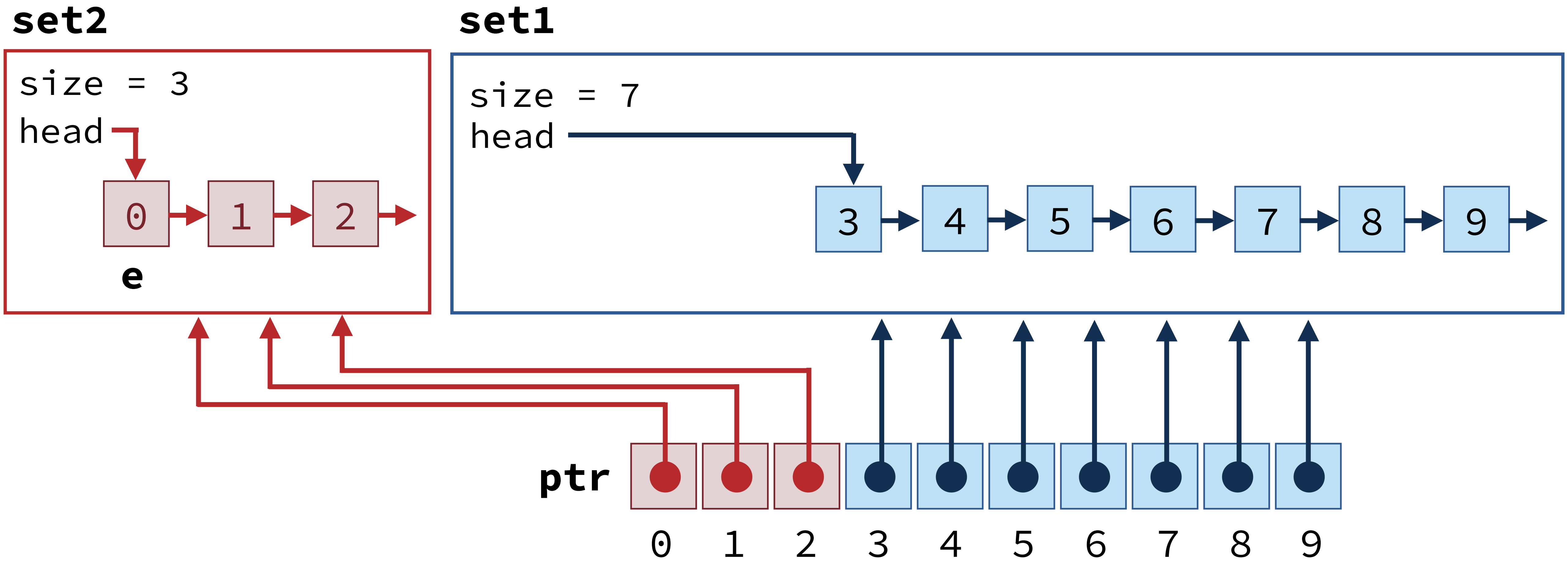

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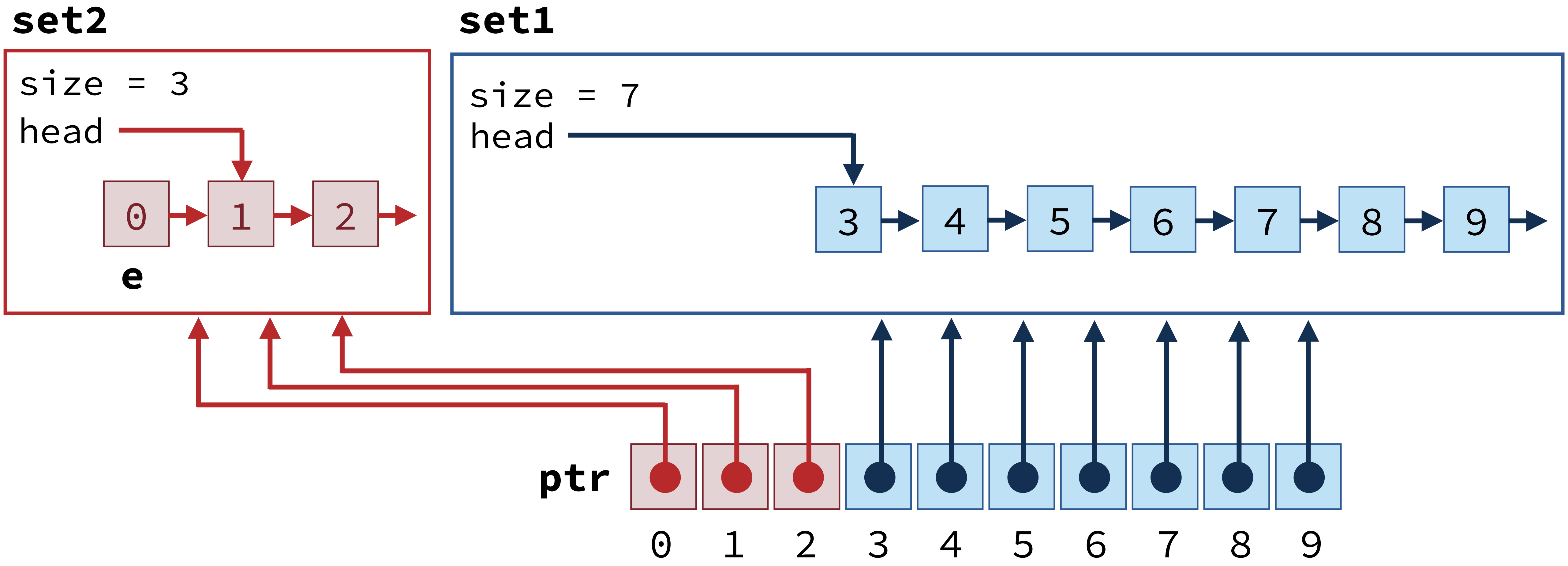

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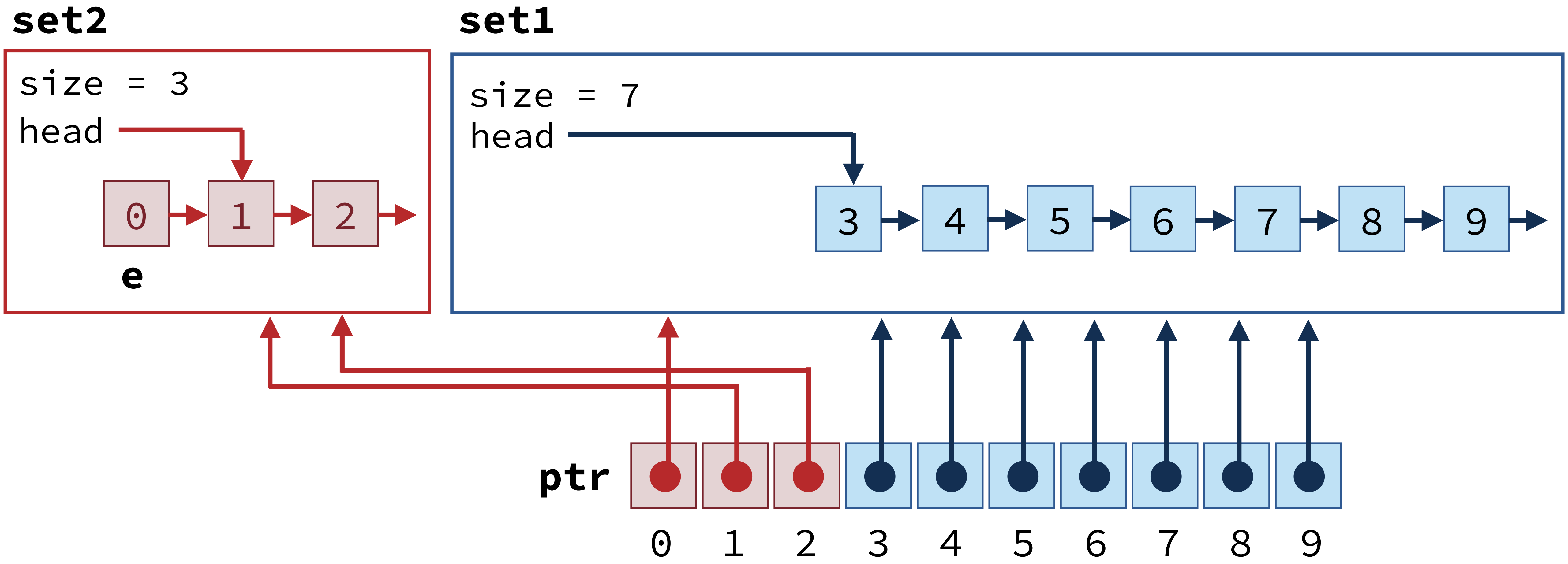

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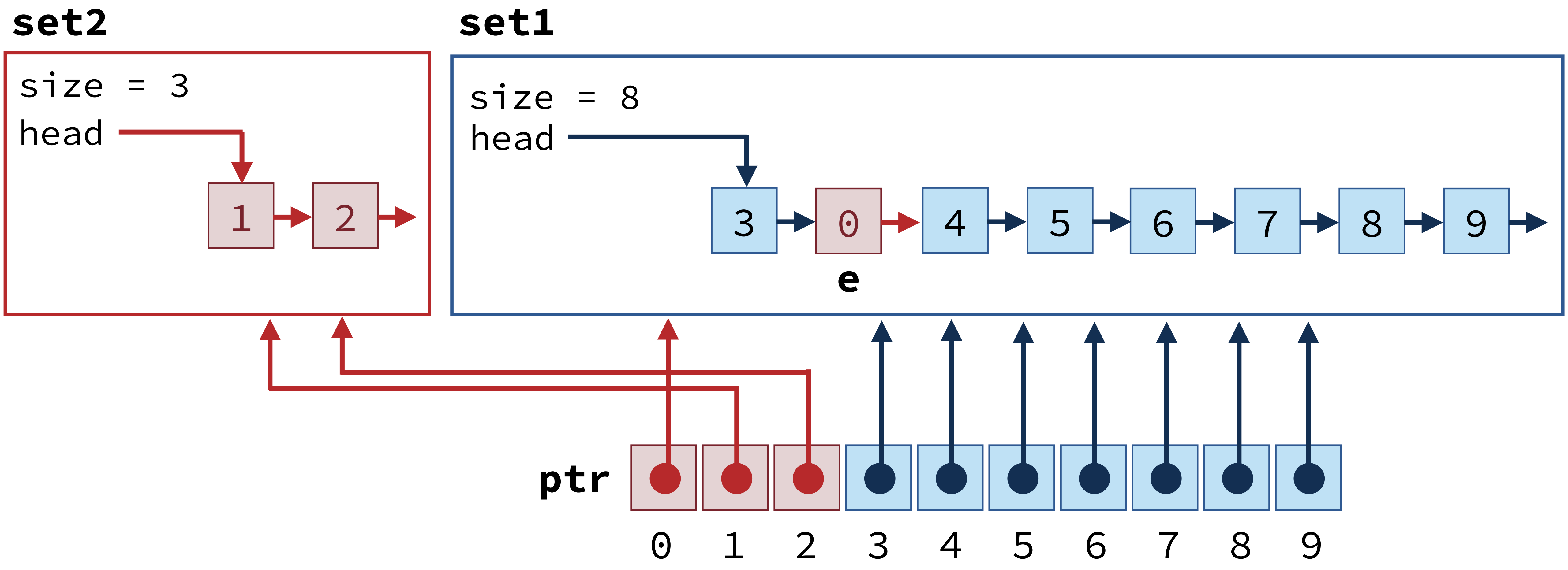

Link-List-Based Quick-Find: `UNION` Example

Example. **`UNION`**`(1, 7)`

1. Make `set2` point at the smaller set and `set1` at the larger set.
2. For each element `e` in `set2:`

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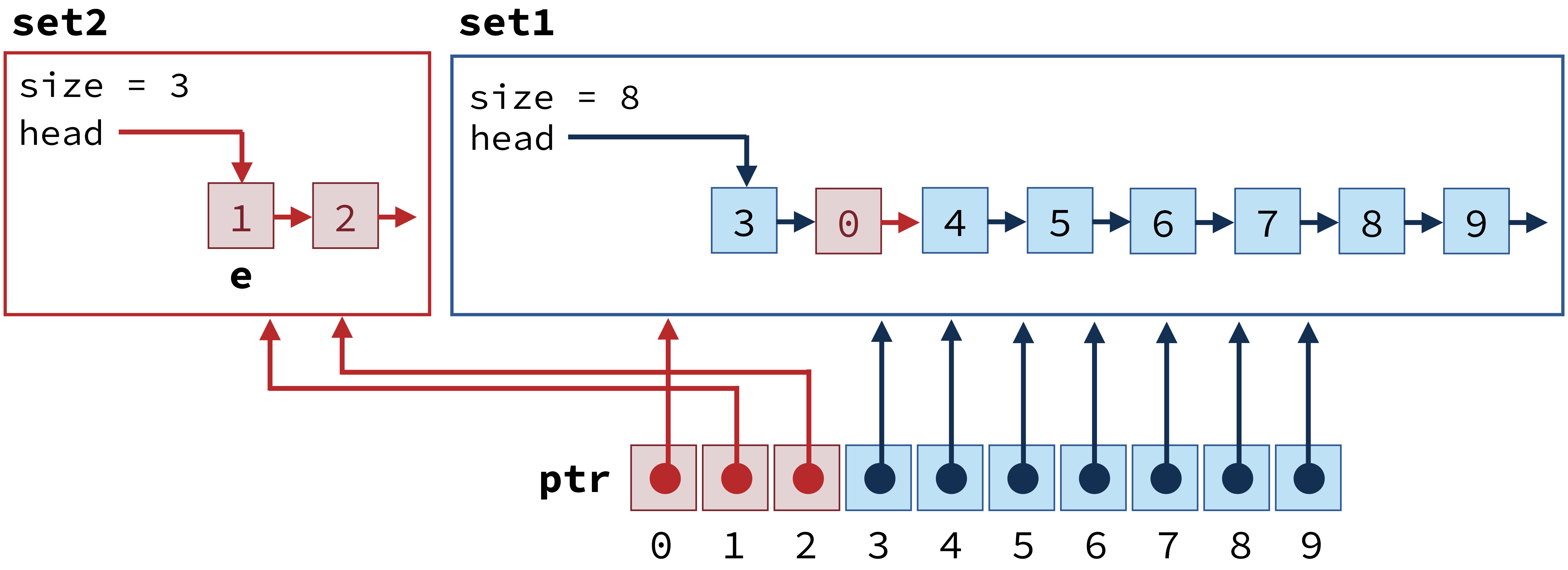

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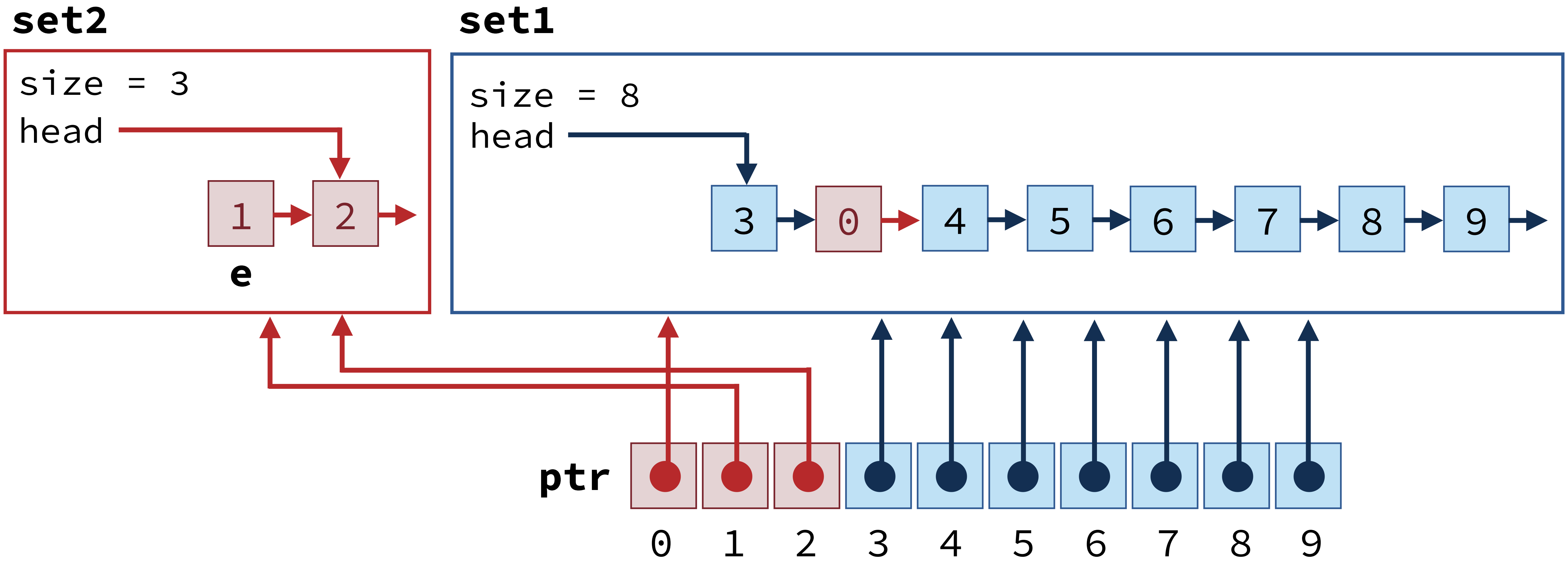

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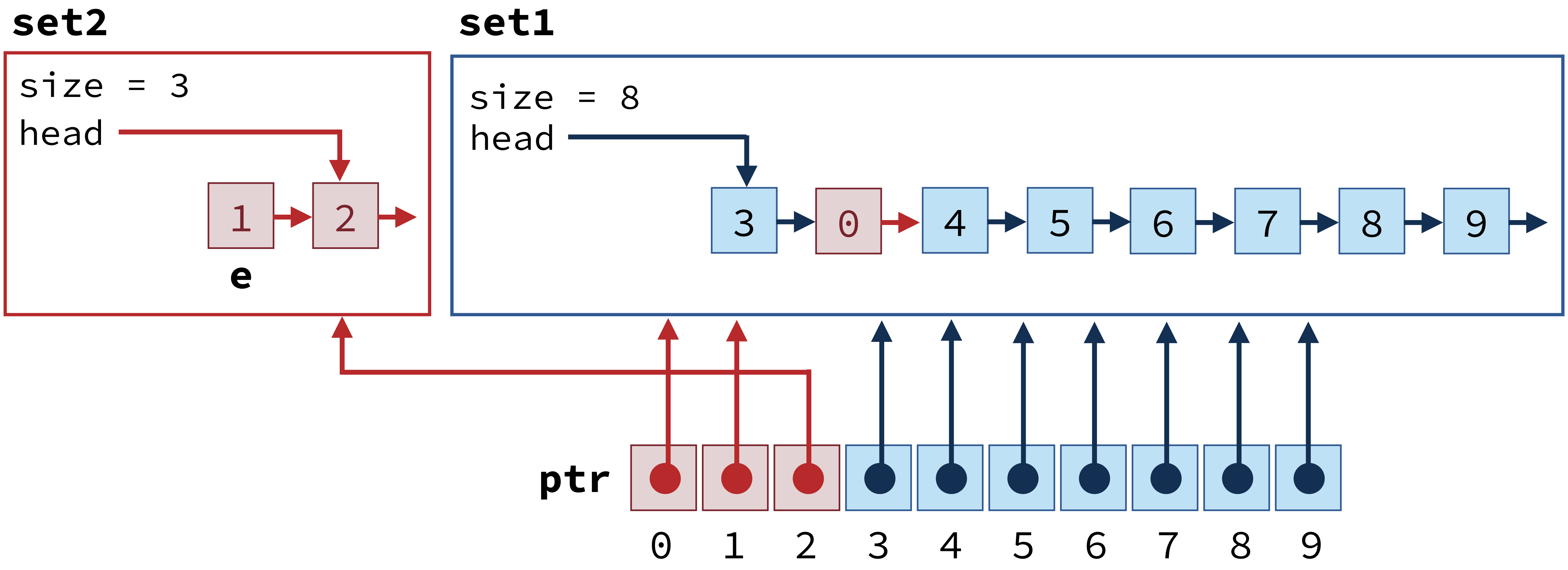

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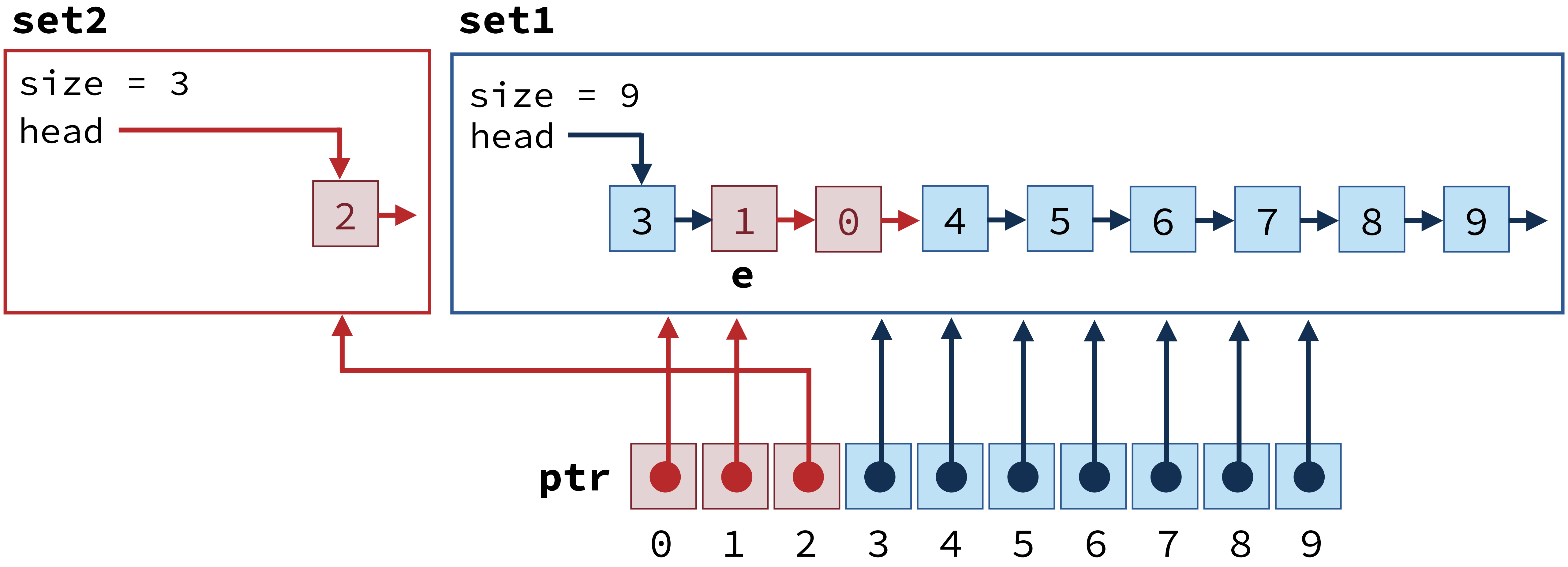

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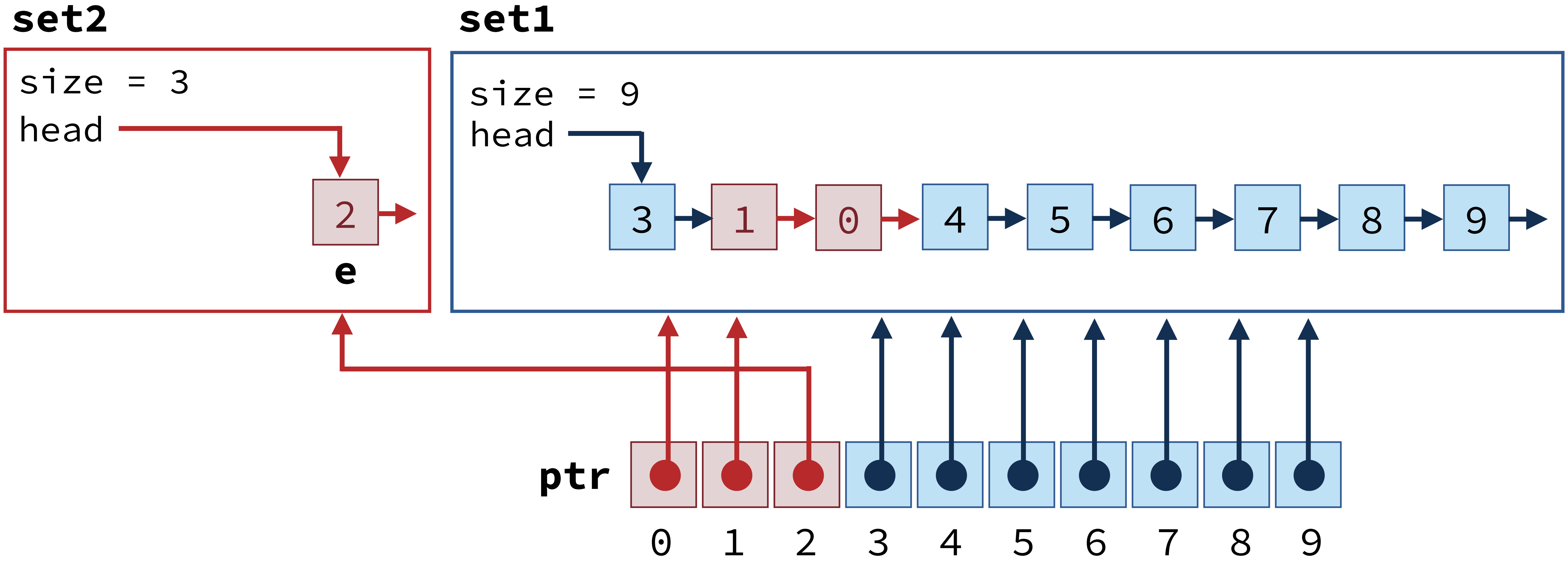

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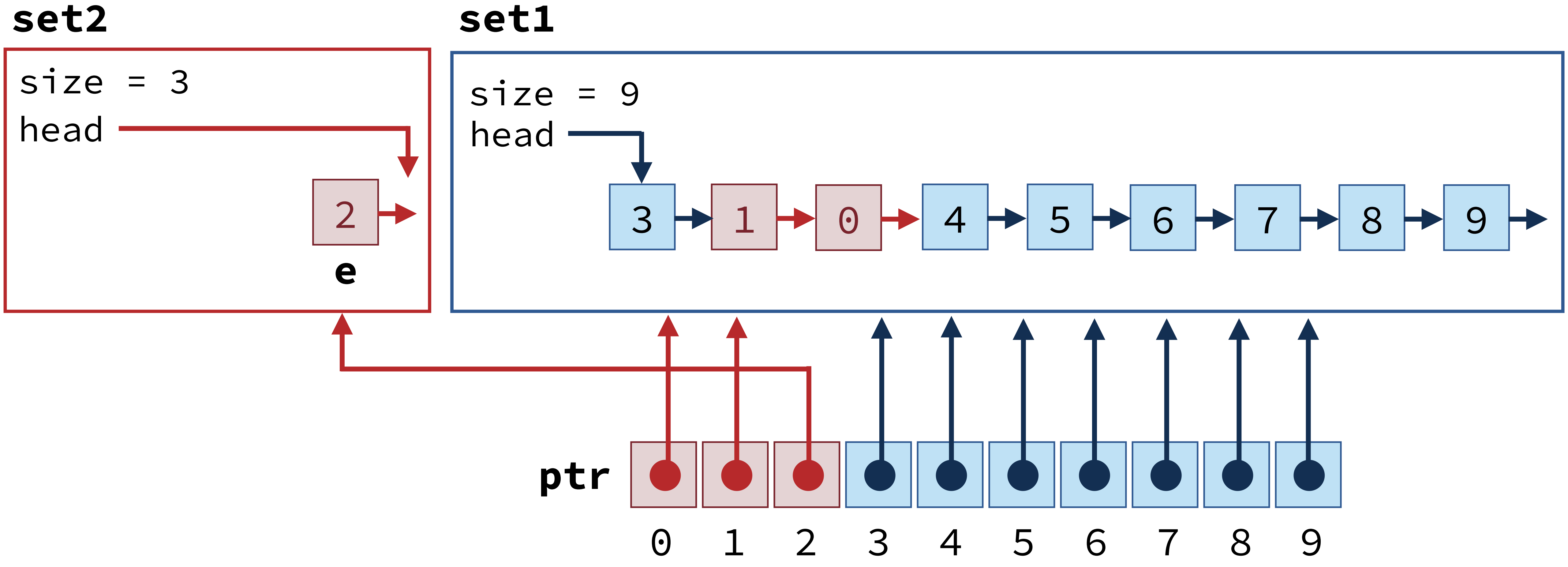

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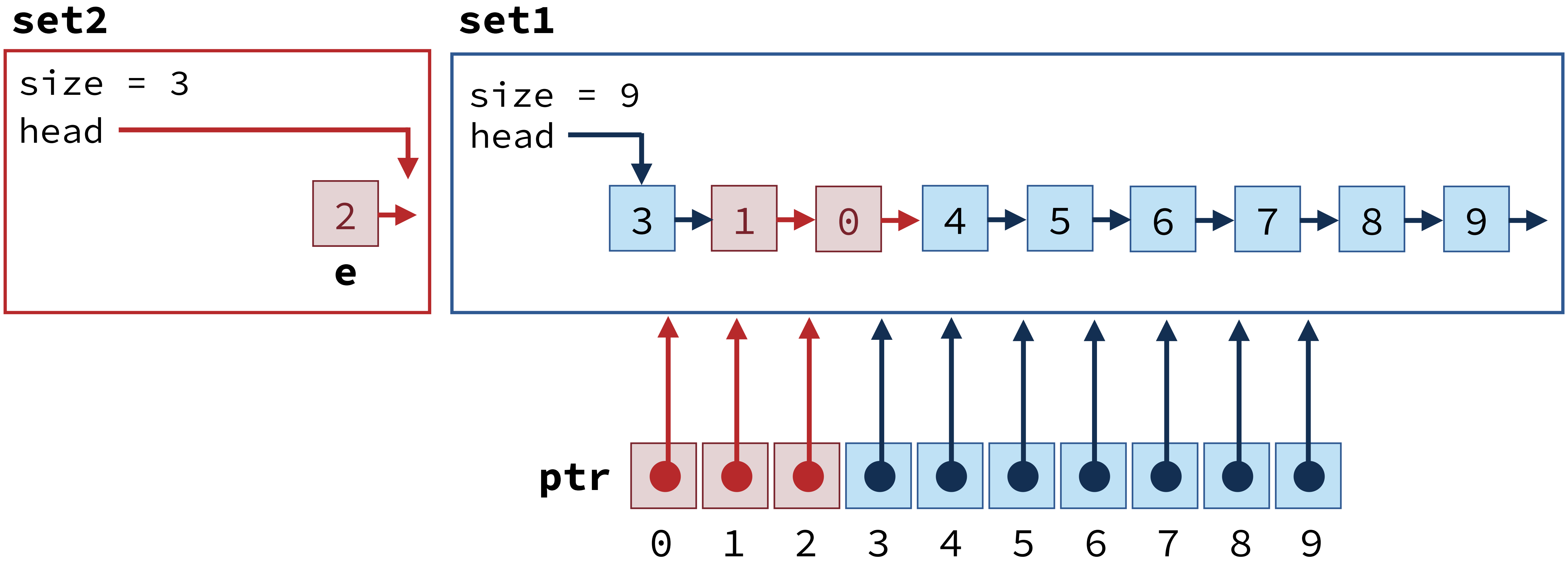

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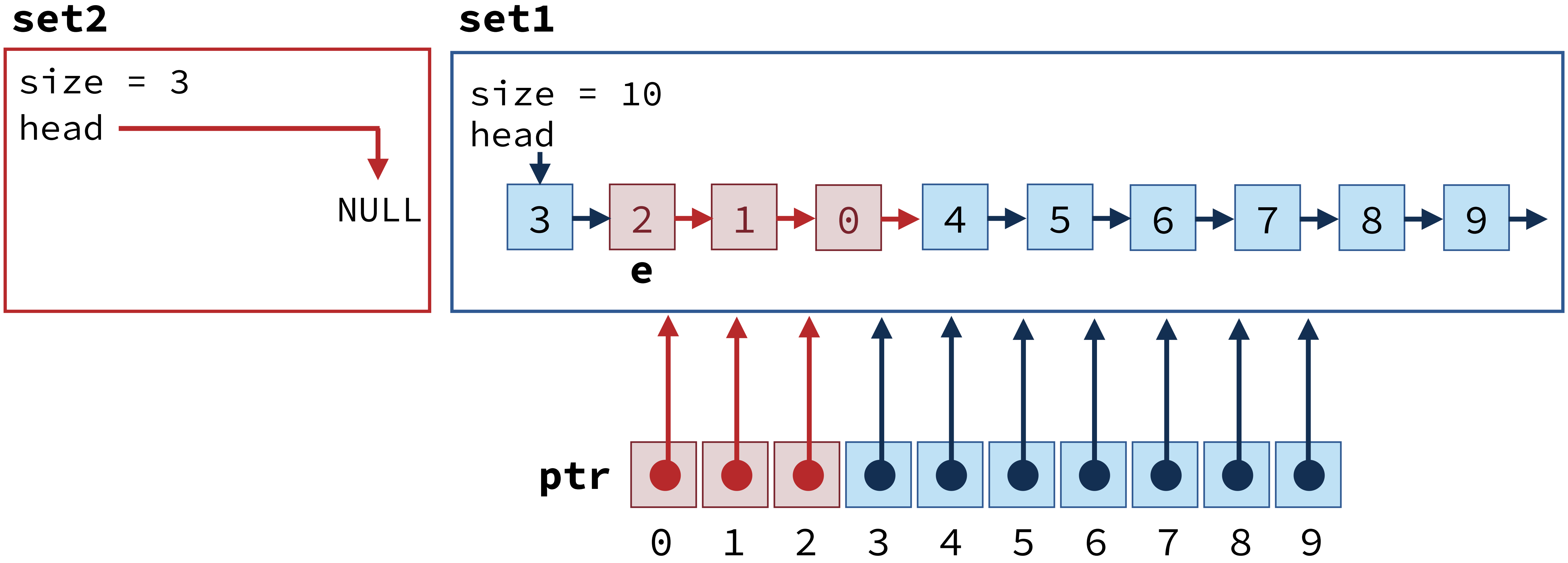

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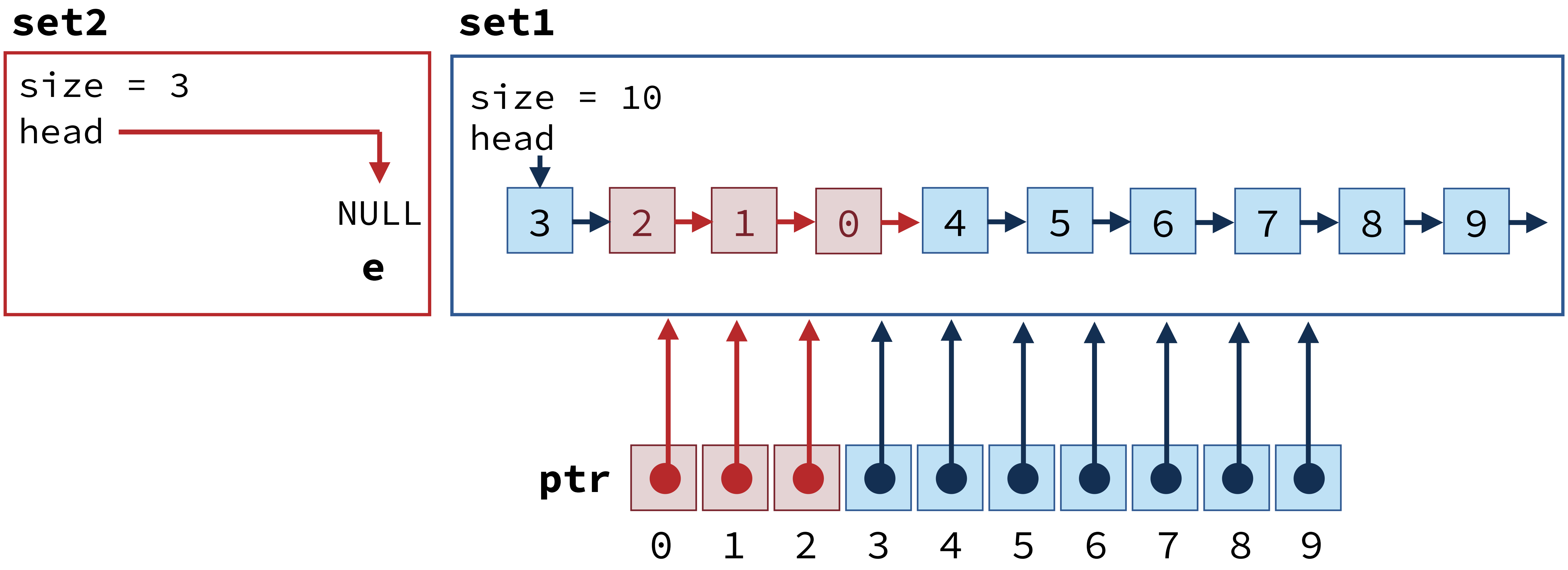
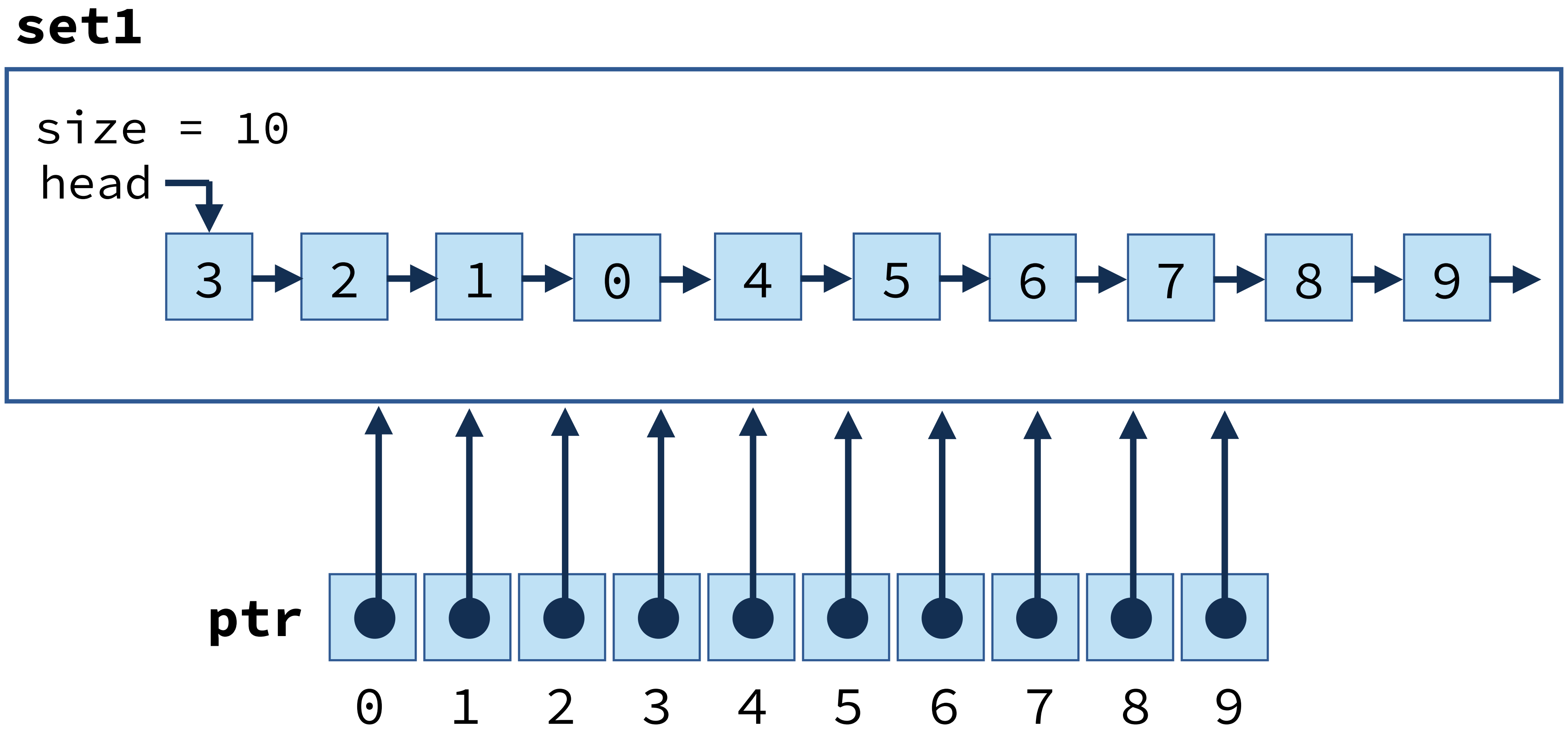


Link-List-Based Quick-Find: `UNION` Example

Example. **`UNION`**`(1, 7)`

```
1. Make set2 point at the smaller set and set1 at the larger set.
2. For each element e in set2:
       ptr[e] = set1
       Move node e to set1 and increment set1.size
3. Remove set2.
```



Link-List-Based Quick-Find: `UNION` Code

```
UNION(p, q)
  if (FIND(p) == FIND(q)) return
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Link-List-Based Quick-Find: UNION Code

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UNION(p, q)
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  LARGE= ptr[p], SMALL = ptr[q]
  if (LARGE.size < SMALL.size)
     SWAP(LARGE, SMALL)
```

Link-List-Based Quick-Find: UNION Code

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LARGE= ptr[p], SMALL = ptr[q]
if (LARGE.size < SMALL.size)
   SWAP(LARGE, SMALL)


// Add into LARGE every element from SMALL
e = SMALL.head
while (e != NULL)
    ptr[e.val]       = LARGE

    SMALL.head       = SMALL.head.next
    e.next           = LARGE.head.next
    LARGE.head.next = e
    LARGE.size     += 1

    e                = SMALL.head

delete SMALL
```

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Worst Case Running Time.

FIND:
UNION:

Cost Model. Number of pointer updates or reads

Link-List-Based Quick-Find: **UNION** Code

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```

Worst Case Running Time.

FIND: $\Theta(1)$

UNION: $\Theta(\min(size_1, size_2))$

Cost Model. Number of pointer updates or reads

Quiz # 1

What is the total running time of a sequence of n `UNION` operations performed on n singleton sets?

Cost Model. Count the number of *pointer reads or updates* .

Choose the *best* answer.

A. $O(n^2)$

B. $O(n \log n)$

C. $O(n)$

D. I can see where this is going …

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What is the total running time of a sequence of n **`UNION`** operations performed on n singleton sets?

Cost Model. Count the number of *pointer reads or updates* .

Choose the *best* answer.

A. $O(n^2)$ correct but too pessimistic!

😁 $O(n \log n)$ correct and tight bound! … *why*?

C. $O(n)$ incorrect! Showing a counterexample is easy.

D. I can see where this is going …

Link-List-Based Quick-Find: Running Time Proof

Observation 1. `ptr[e]` changes during a **`UNION`** operation only if **`e`** is in the smaller set.

Observation 2. If `ptr[e]` changes because of a **`UNION`** operation, **`e`** becomes in a set whose size is at least double the size it was in before the **`UNION`** operation.

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From observation 2, this means that the size of the set containing `e` at least doubled more than $\log_2(n)$ times, which implies that e is in a set whose size is $> 2^{\log_2(n)} > n$, which is impossible because there are only n elements.

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Proposition. **`UNION`** runs in $O(\log n)$ amortized time.

Proof. In a sequence of n **`UNION`** operations, no pointer can change more than $\log_2(n)$ times in total. Hence, the total is $O(n \log n)$ and each **`UNION`** operation costs $O(\log_2 n)$ on average.

Quiz # 2

Provide a sequence of `UNION` operations that leads to a running time of $\Theta(n)$ and another one that leads to a running time of $\Theta(n \log n)$.

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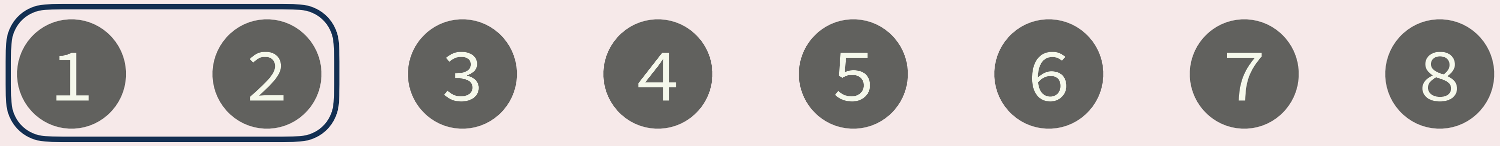

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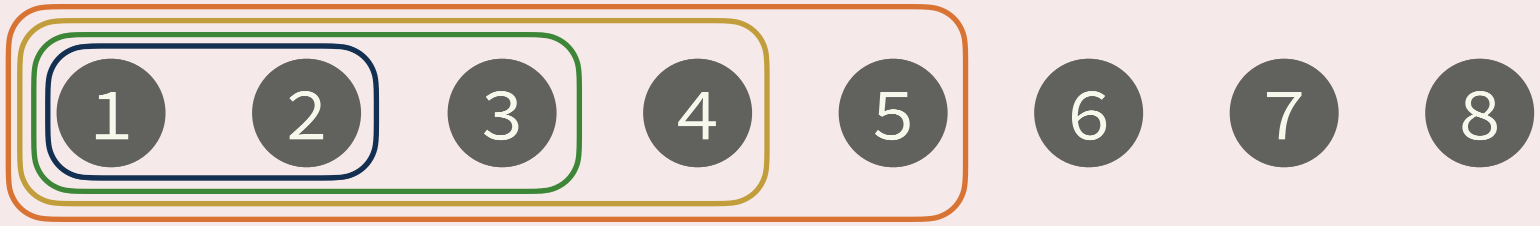

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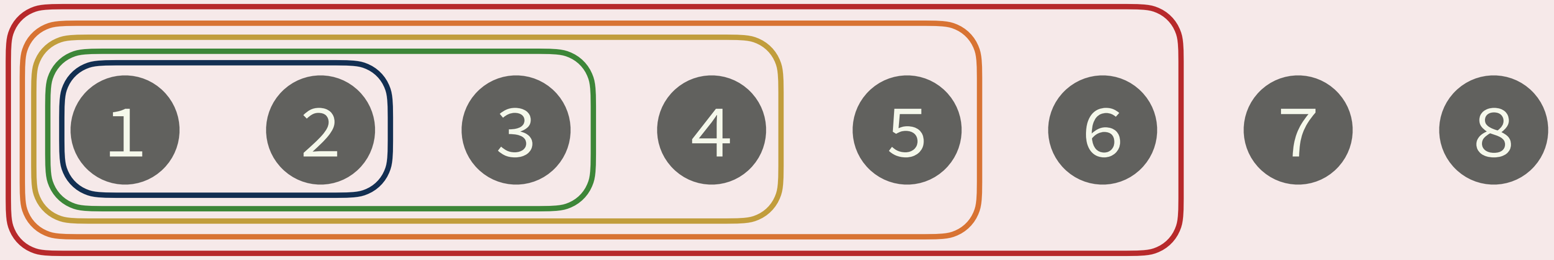

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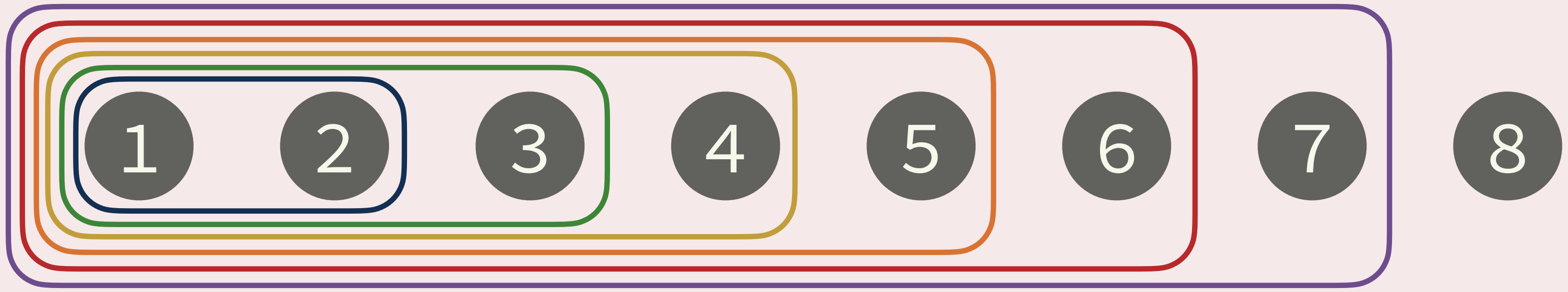

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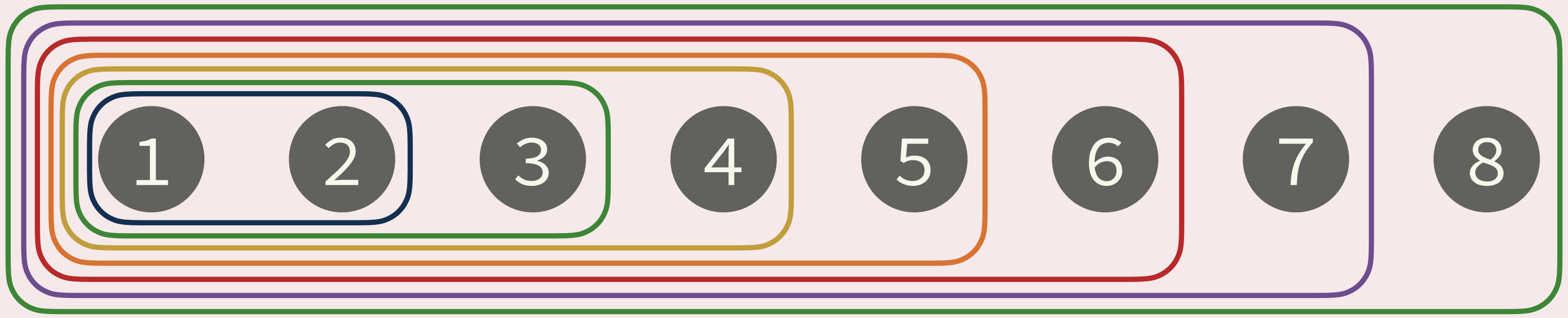

$$(n-1) \times 1 = O(n)$$

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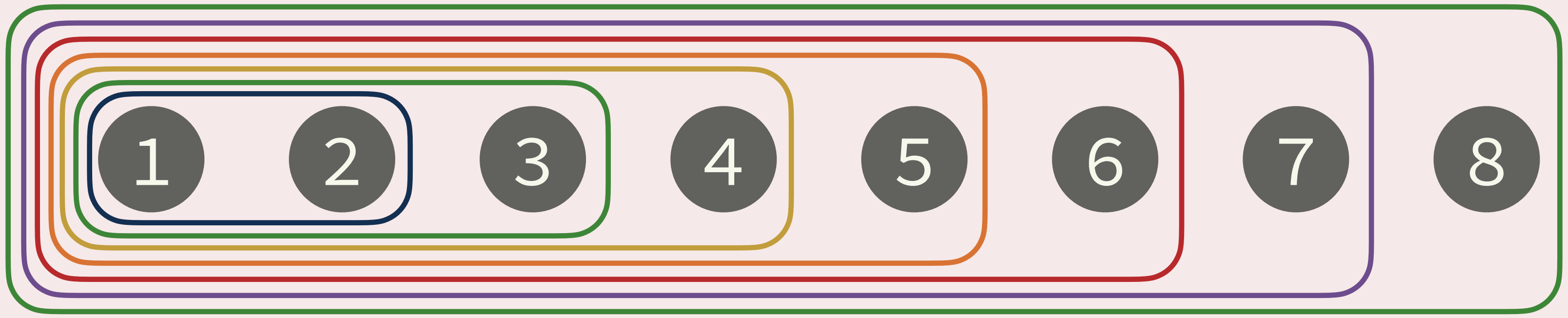

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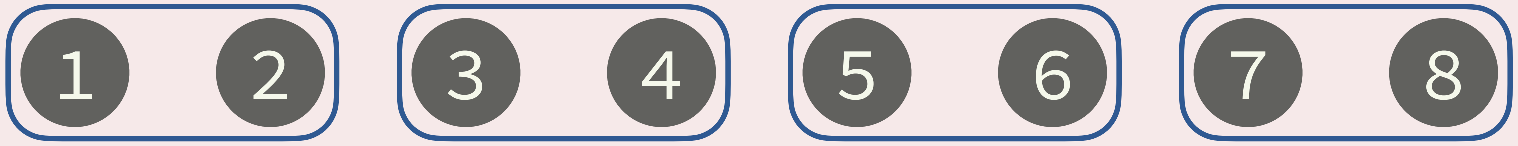

$$\frac{n}{2^1} \times 2^0$$

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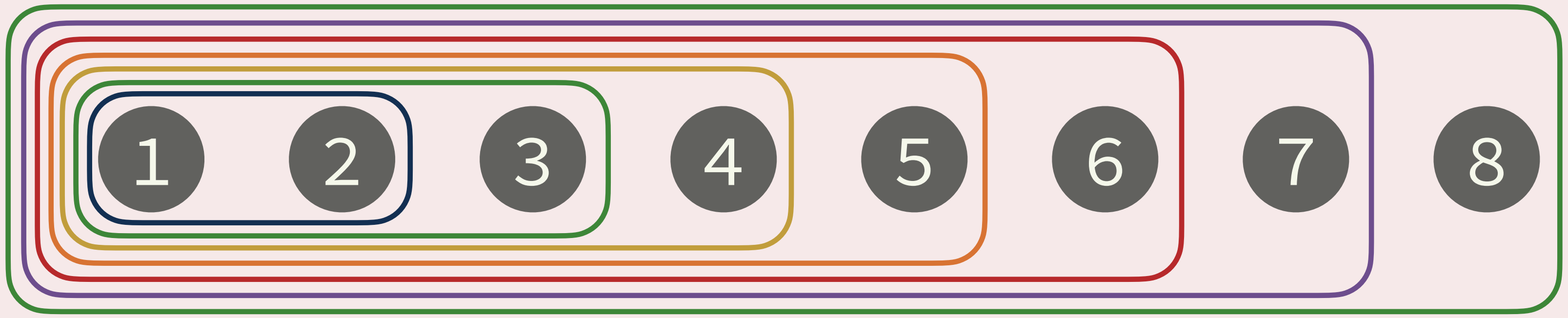

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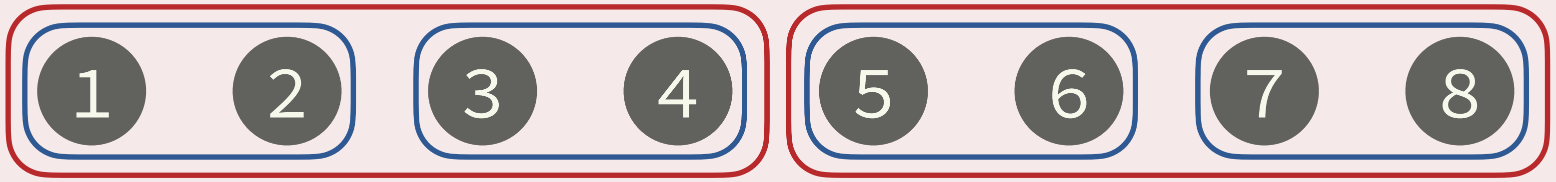

$$\frac{n}{2^1} \times 2^0 \quad + \quad \frac{n}{2^2} \times 2^1$$

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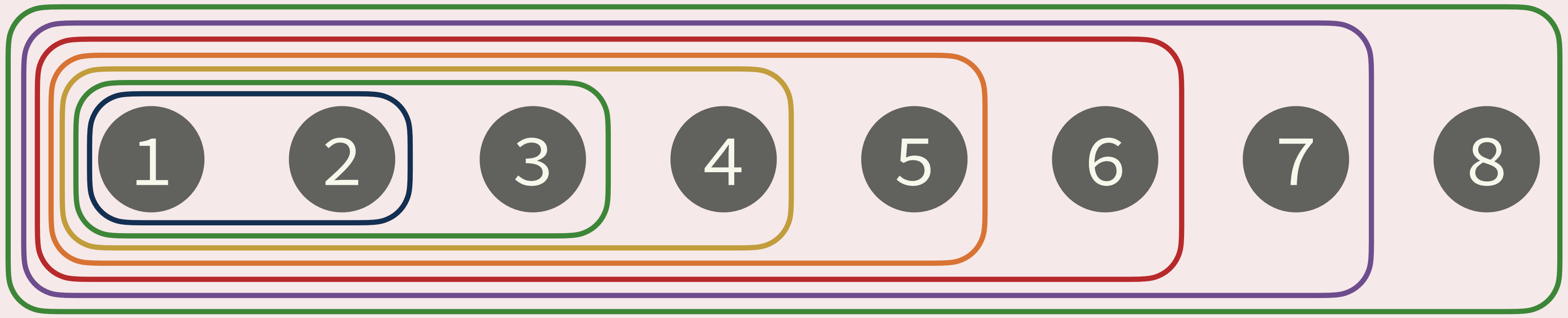

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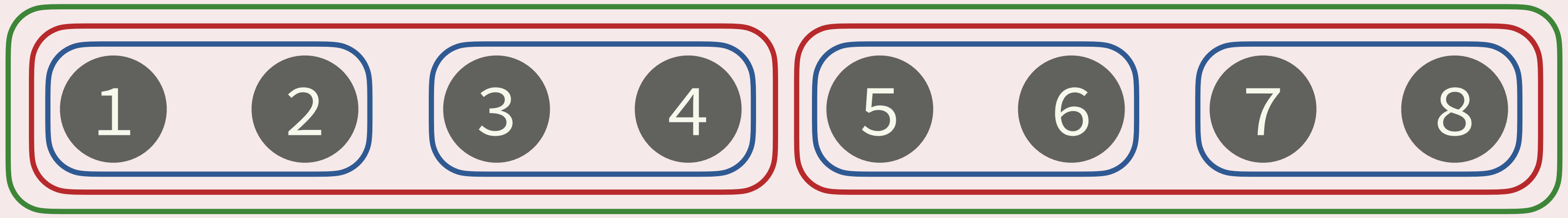

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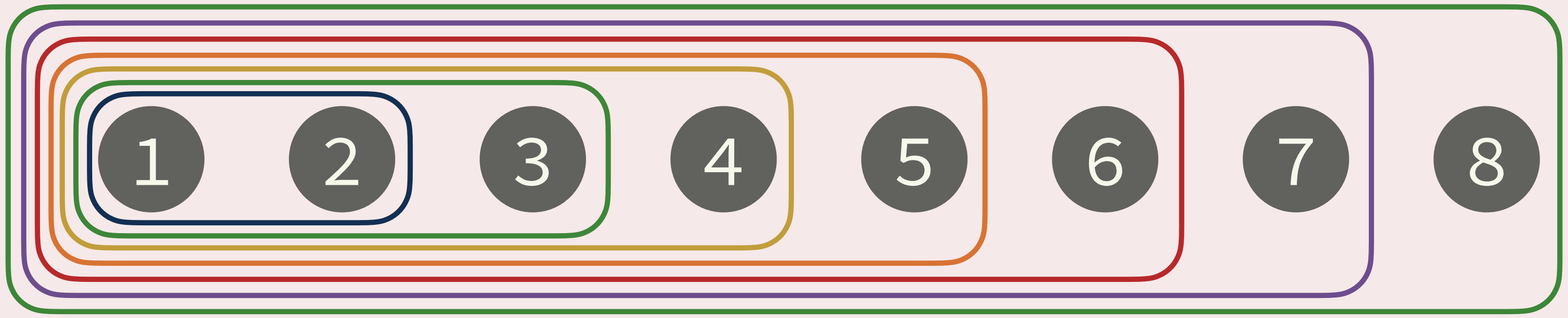

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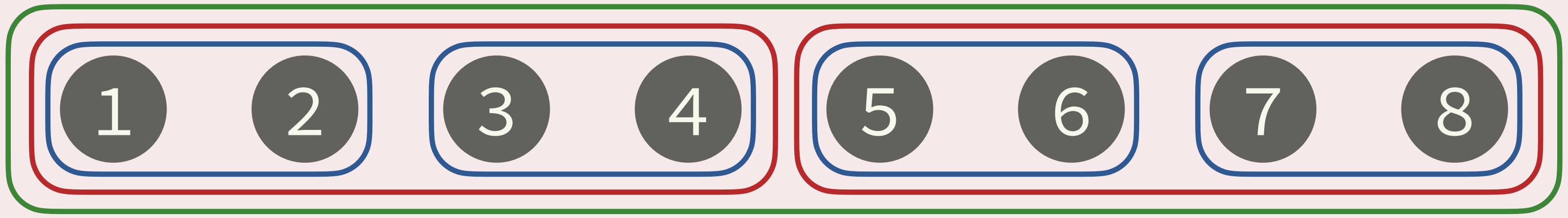

$$\frac{n}{2^1} \times 2^0 + \frac{n}{2^2} \times 2^1 + \frac{n}{2^3} \times 2^2 = \frac{n}{2} \times \log_2(n)$$

Quick-Find: Running Time Summary

	Array-based Quick-Find	Linked-List-based Quick-Find
`FIND`	$O(1)$	$O(1)$
`UNION`	$O(n)$	$O(n)$
Sequence of n `UNION` operations:		

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🤔

Can we do better?

Improvement Attempt 2: Quick-Union

Idea. Each set has a *canonical element* (root, representative or leader for the set.)

- **`FIND`**`(p)` Returns the root of the set of `p`.
- **`UNION`**`(p, q)` Change the root of the set of `q` to be the result of **`FIND`**`(p)`.

Initially. Every element `e` is in a singleton set whose root is `e` itself.

Example.

`parent``[]`

0	1	2	3	4	5	6	7	8	9
0	1	2	3	4	5	6	7	8	9

- **`UNION`**`(0, 3)`
- **`UNION`**`(3, 1)`
- **`UNION`**`(4, 1)`
- **`UNION`**`(7, 5)`
- **`UNION`**`(1, 5)`
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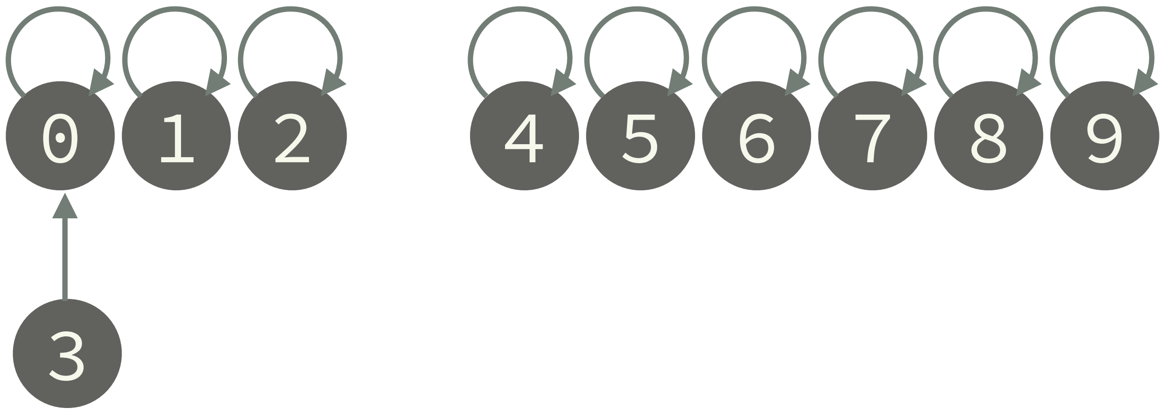

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- **`FIND`**`(p)` Returns the root of the set of `p`.
- **`UNION`**`(p, q)` Change the root of the set of `q` to be the result of **`FIND`**`(p)`.

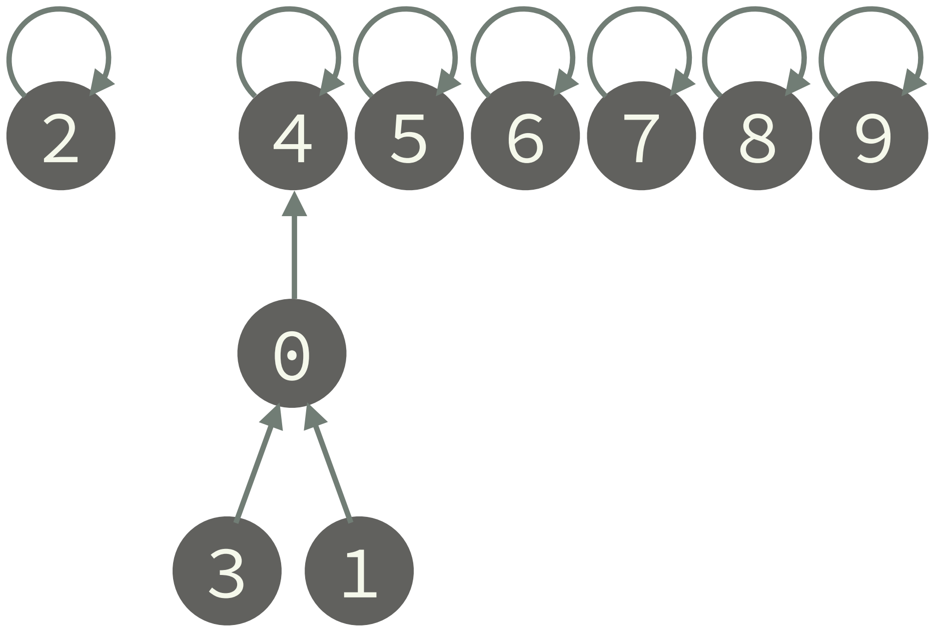
Initially. Every element `e` is in a singleton set whose root is `e` itself.

Example.

`parent[]`

4	0	2	0	4	5	6	7	8	9
0	1	2	3	4	5	6	7	8	9

- `UNION(0, 3)`
- `UNION(3, 1)`
- **`UNION(4, 1)`**
- `UNION(7, 5)`
- `UNION(1, 5)`
- `UNION(2, 7)`
- `UNION(0, 9)`
- `UNION(6, 1)`



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Example.

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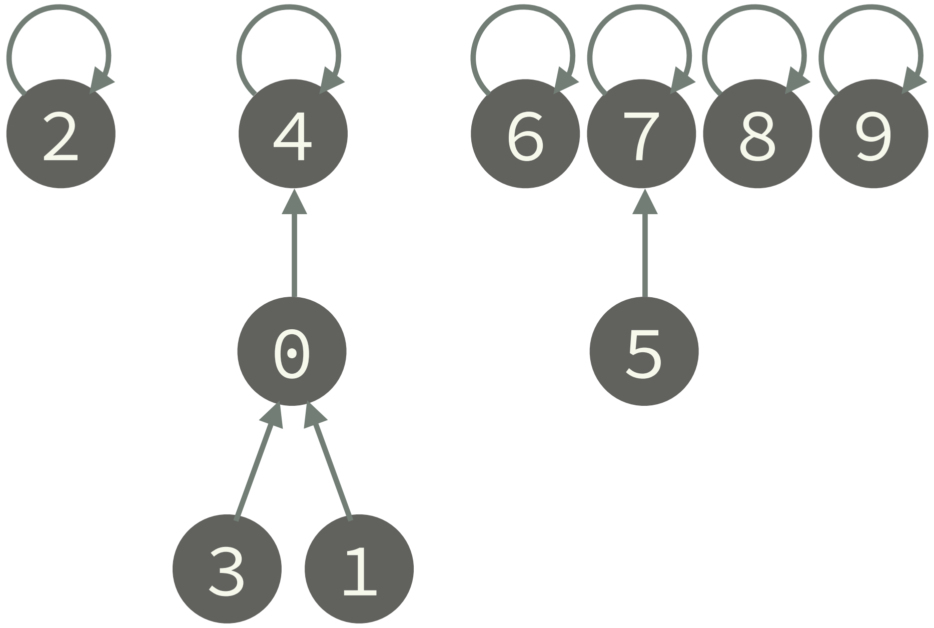
4	0	2	0	4	7	6	7	8	9
0	1	2	3	4	5	6	7	8	9



- UNION(0, 3)
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Example.

`parent``[]`

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0	1	2	3	4	5	6	7	8	9

- `UNION(0, 3)`
- `UNION(3, 1)`
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- **`UNION(1, 5)`**
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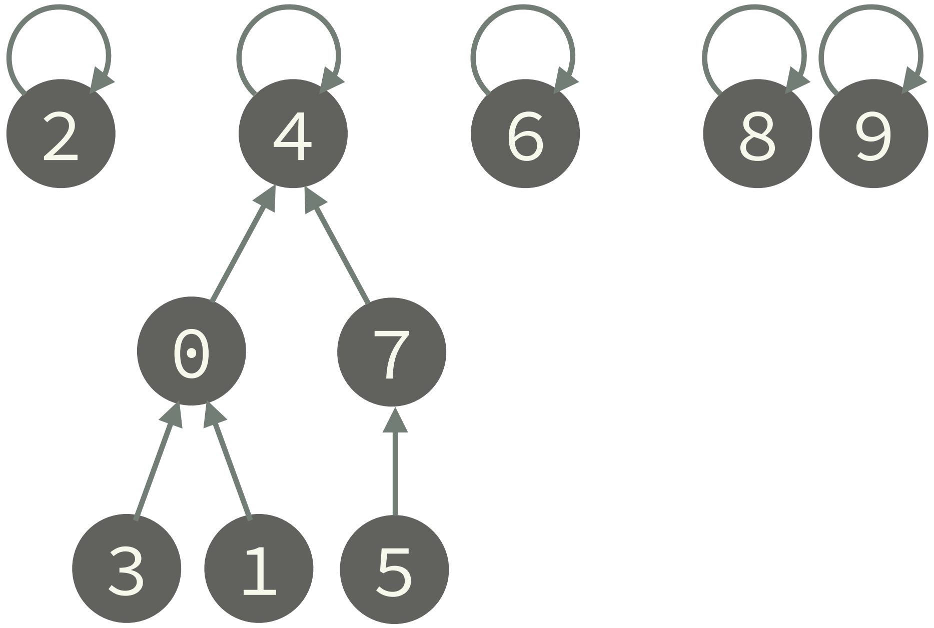

Improvement Attempt 2: Quick-Union

Idea. Each set has a *canonical element* (root, representative or leader for the set.)

- `FIND(p)` Returns the root of the set of `p`.
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Example.

parent[]

4	0	2	0	2	7	6	4	8	9
0	1	2	3	4	5	6	7	8	9

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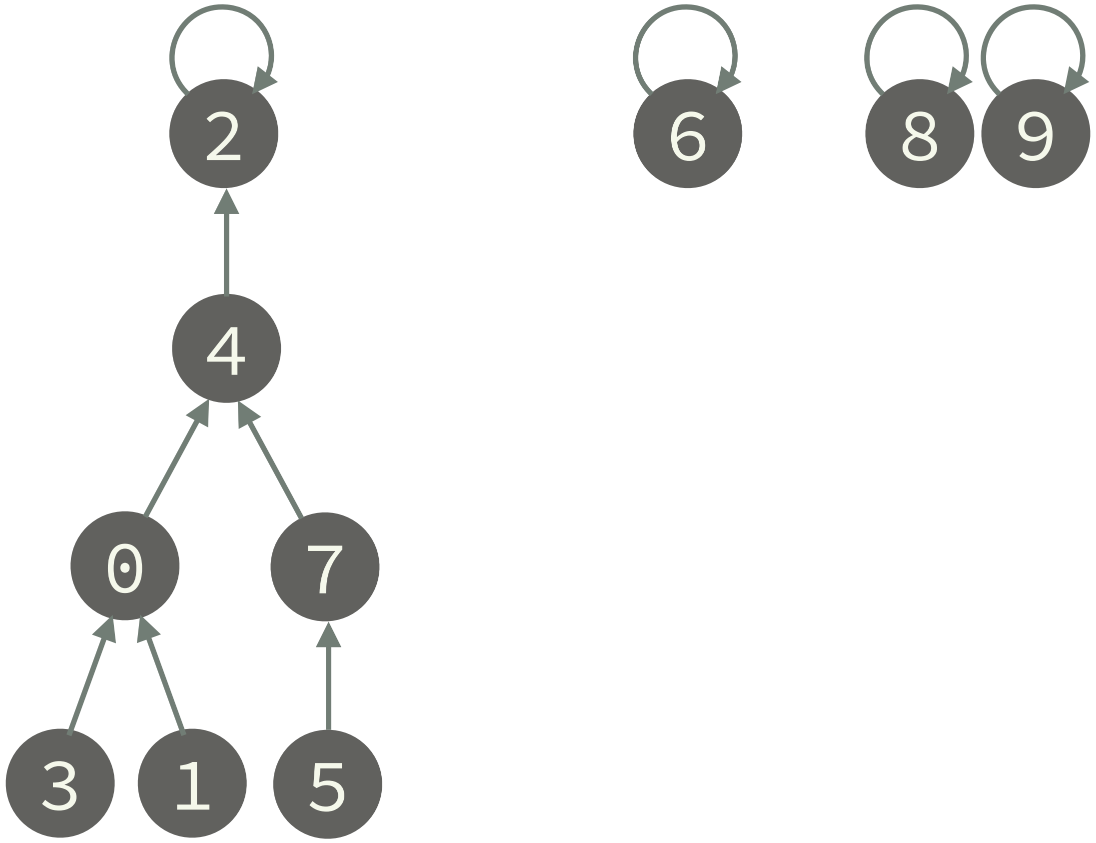

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Example.

`parent``[]`

4	0	2	0	2	7	6	4	8	**2**
0	1	2	3	4	5	6	7	8	9

- `UNION(0, 3)`
- `UNION(3, 1)`
- `UNION(4, 1)`
- `UNION(7, 5)`
- `UNION(1, 5)`
- `UNION(2, 7)`
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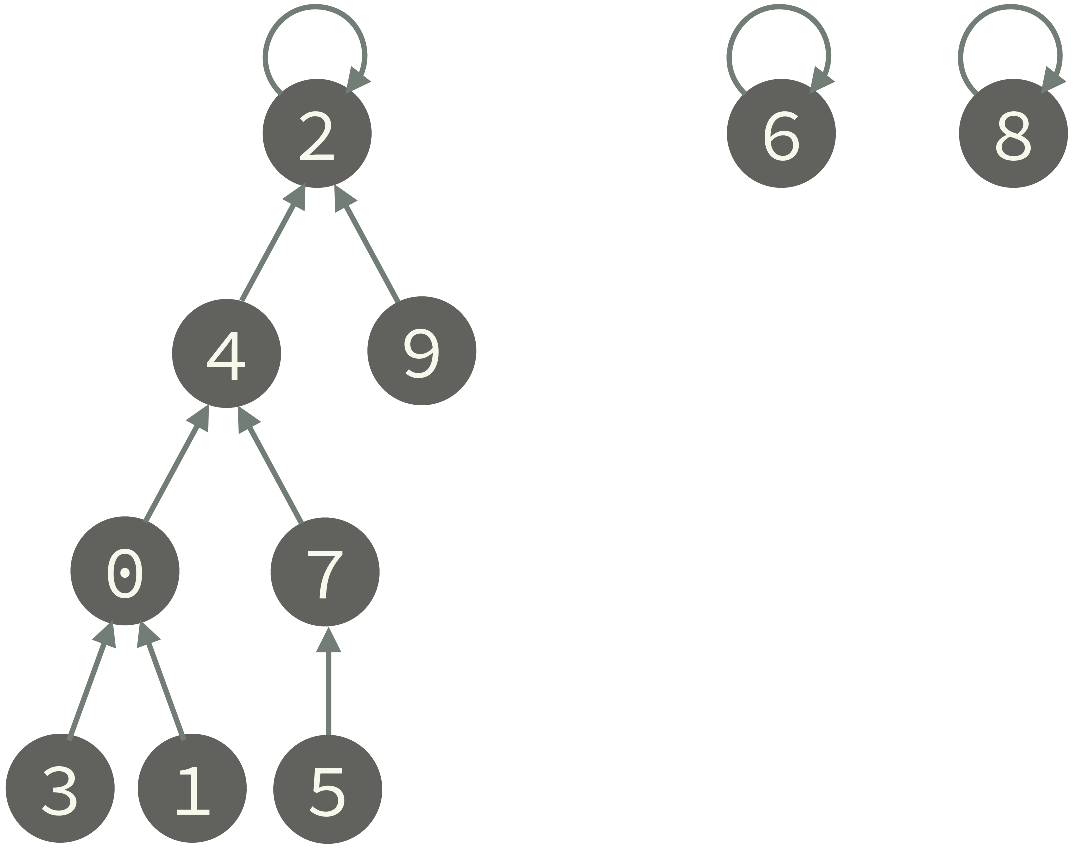

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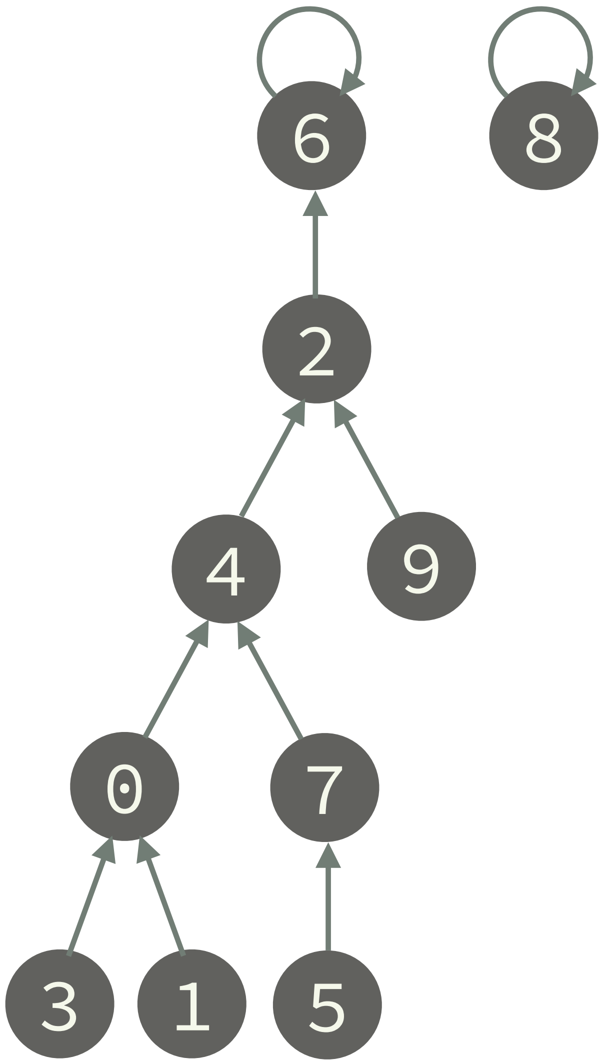
Initially. Every element `e` is in a singleton set whose root is `e` itself.

Example.

`parent``[]`

4	6	2	0	2	7	6	4	8	2
0	1	2	3	4	5	6	7	8	9

- `UNION(0, 3)`
- `UNION(3, 1)`
- `UNION(4, 1)`
- `UNION(7, 5)`
- `UNION(1, 5)`
- `UNION(2, 7)`
- `UNION(0, 9)`
- **`UNION`**`(6, 1)`



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- `FIND(p)` Returns the root of the set of `p`.
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Initially. Every element `e` is in a singleton set whose root is `e` itself.

Example.

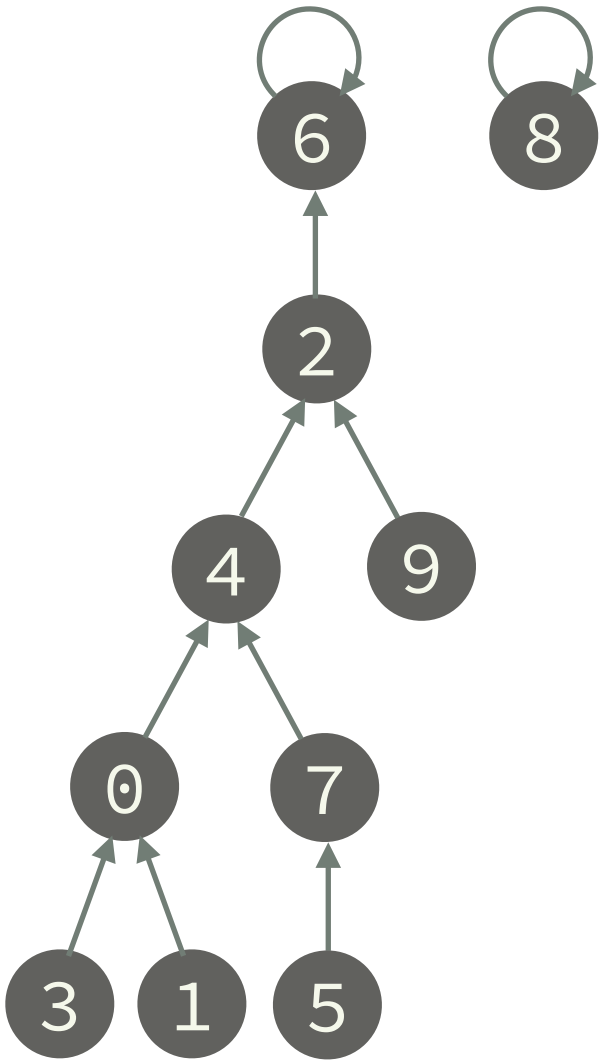
parent[]

4	6	2	0	2	7	6	4	8	2
0	1	2	3	4	5	6	7	8	9

UNION(p, q)

```
root1 = FIND(p)
root2 = FIND(q)

if (root1 != root2)
   parent[root2] = root1
```



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Example.

parent[]

4	6	2	0	2	7	6	4	8	2
0	1	2	3	4	5	6	7	8	9

UNION(p, q)

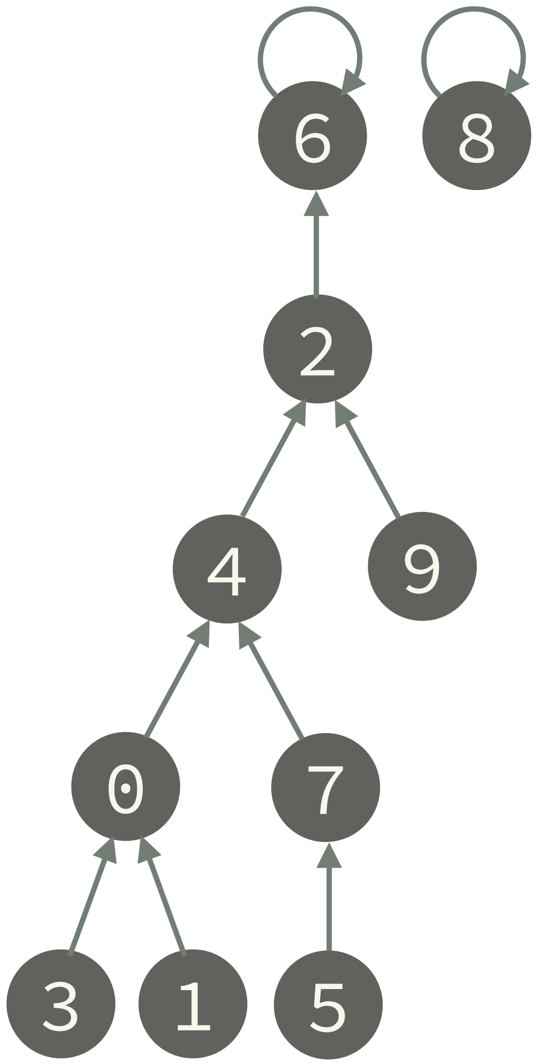
```
root1 = FIND(p)
root2 = FIND(q)

if (root1 != root2)
   parent[root2] = root1
```

FIND(p)

```
while (parent[p] != p)
    p = parent[p]

return p
```



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- **`FIND`**`(p)` Returns the root of the set of `p`.
- **`UNION`**`(p, q)` Change the root of the set of `q` to be the result of **`FIND`**`(p)`.

Initially. Every element `e` is in a singleton set whose root is `e` itself.

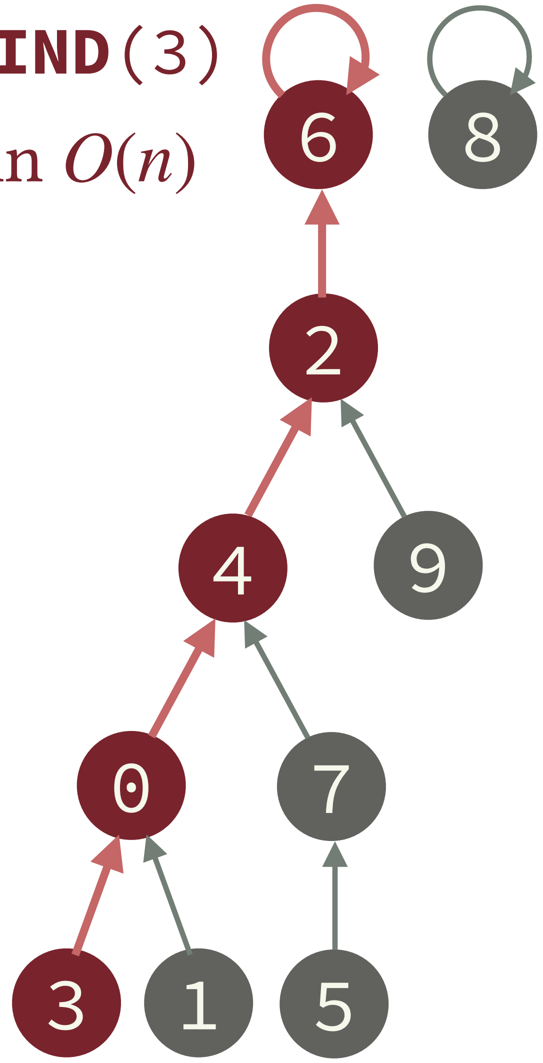
Example.

`parent``[]`

0	1	2	3	4	5	6	7	8	9
4	6	2	0	2	7	6	4	8	2

`FIND``(3)`

Runs in $O(n)$



`UNION``(p, q)`

```
root1 = FIND(p)
root2 = FIND(q)

if (root1 != root2)
   parent[root2] = root1
```

`FIND``(p)`

```
while (parent[p] != p)
    p = parent[p]

return p
```

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Example.

parent[]

4	6	2	0	2	7	6	4	8	2
0	1	2	3	4	5	6	7	8	9

UNION(p, q)

```
root1 = FIND(p)
root2 = FIND(q)

if (root1 != root2)
   parent[root2] = root1
```

FIND(p)

```
while (parent[p] != p)
    p = parent[p]

return p
```

Worst Case Running Time.

FIND: $O(n)$
UNION: $O(n)$

Cost Model. Number of array accesses.

Improvement Attempt 2.1: Weighted Quick-Union (by size)

Idea. Attach the smaller tree to the larger tree.

Rationale. Reduce the likelihood of long chains.

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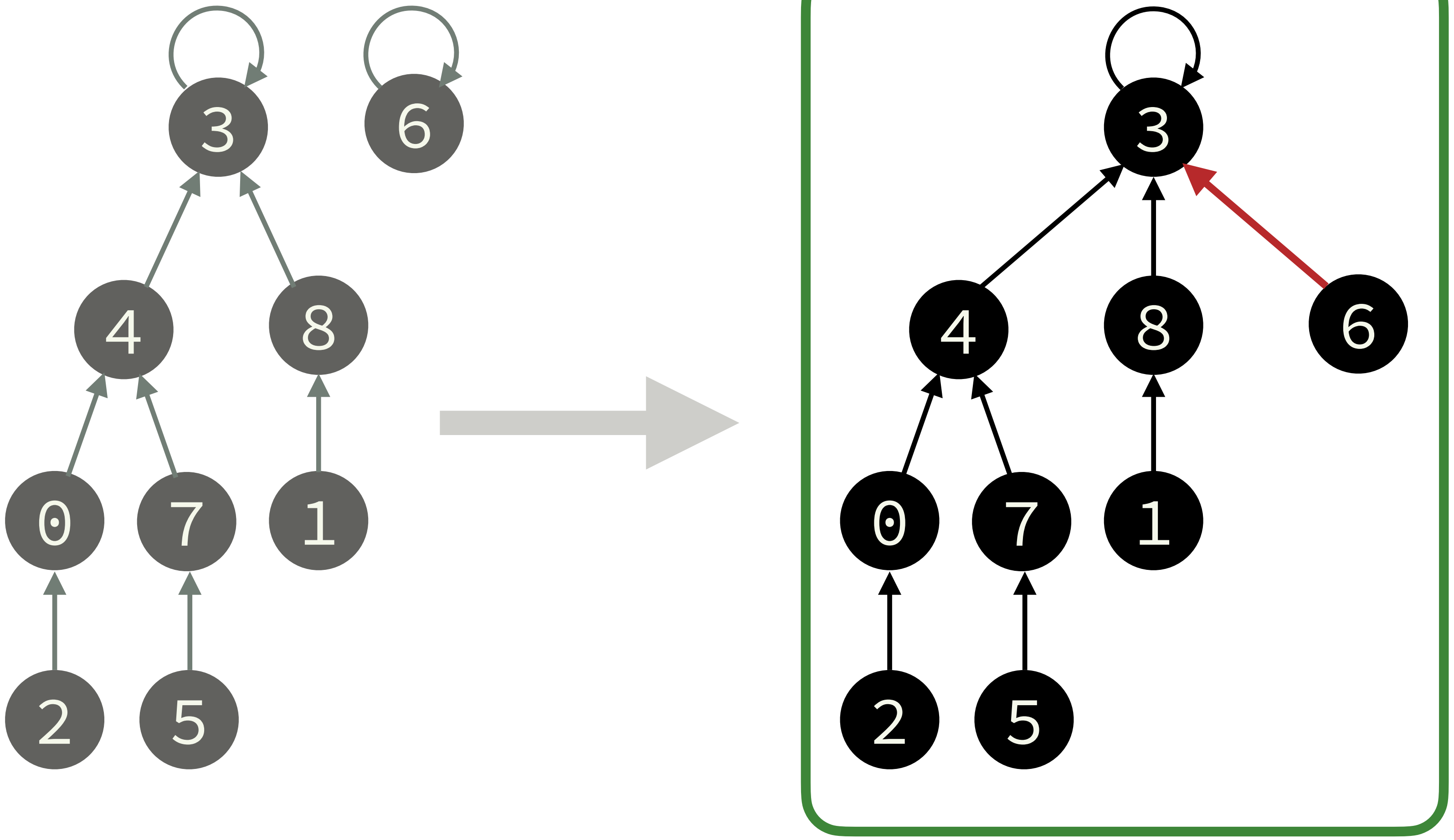
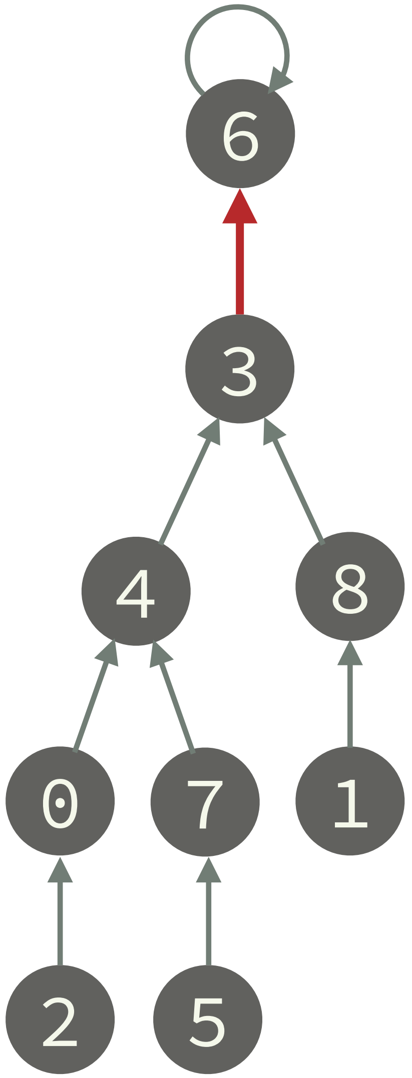
Rationale. Reduce the likelihood of long chains.

Example. **`UNION`**`(6, 2)` attaches 6 ⟶ 3 not 3 ⟶ 6.



did not increase the height



increased the height

Improvement Attempt 2.1: Weighted Quick-Union (by size)

Idea. Attach the smaller tree to the larger tree.

Rationale. Reduce the likelihood of long chains.

Modification. Add an array to record the size of the tree rooted at each element.

parent	0	1	2	3	4	5	6	7	8
size	1	1	1	1	1	1	1	1	1
	0	1	2	3	4	5	6	7	8

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size	1	1	1	1	1	1	1	1	1
	0	1	2	3	4	5	6	7	8

UNION(p, q)

```
r1 = FIND(p)
r2 = FIND(q)

if (r1 == r2) return
```

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UNION(p, q)

```
r1 = FIND(p)
r2 = FIND(q)

if (r1 == r2) return

if (size[r1] < size[r2])
    SWAP(r1, r2)
```

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    SWAP(r1, r2)

parent[r2] = r1
size[r1]  += size[r2]
```

```
UNION(0, 1)
UNION(2, 1)
UNION(4, 3)
UNION(2, 3)
```

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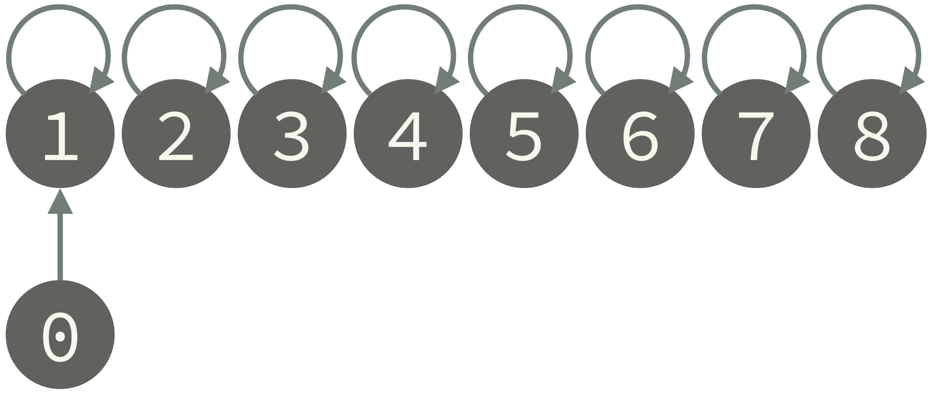
size	1	2	1	1	1	1	1	1	1
	0	1	2	3	4	5	6	7	8



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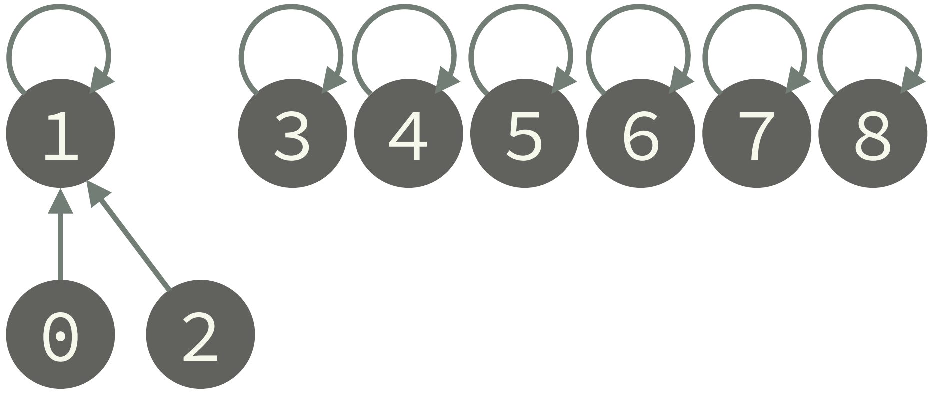
Rationale. Reduce the likelihood of long chains.

Modification. Add an array to record the size of the tree rooted at each element.

	0	1	2	3	4	5	6	7	8
parent	1	1	**1**	3	4	5	6	7	8
size	1	**3**	1	1	1	1	1	1	1



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```

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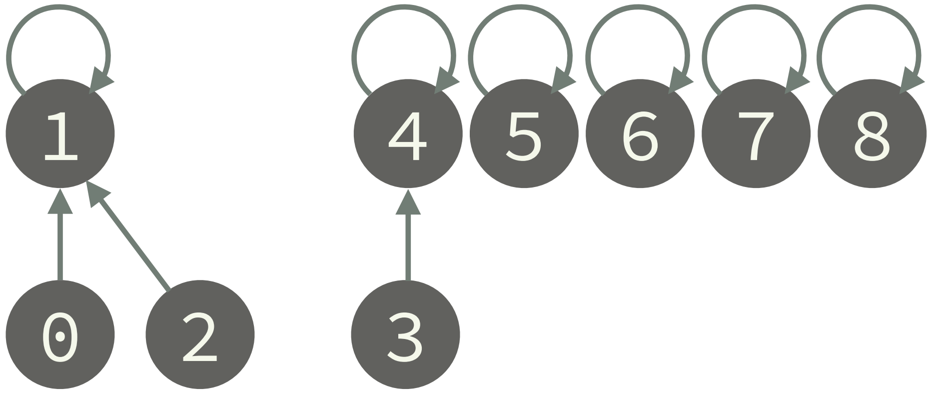
Idea. Attach the smaller tree to the larger tree.

Rationale. Reduce the likelihood of long chains.

Modification. Add an array to record the size of the tree rooted at each element.

parent	1	1	1	**4**	4	5	6	7	8
size	1	3	1	1	**2**	1	1	1	1
	0	1	2	3	4	5	6	7	8



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Rationale. Reduce the likelihood of long chains.

Modification. Add an array to record the size of the tree rooted at each element.

	0	1	2	3	4	5	6	7	8
parent	1	1	1	4	**1**	5	6	7	8
size	1	**5**	1	1	2	1	1	1	1

```
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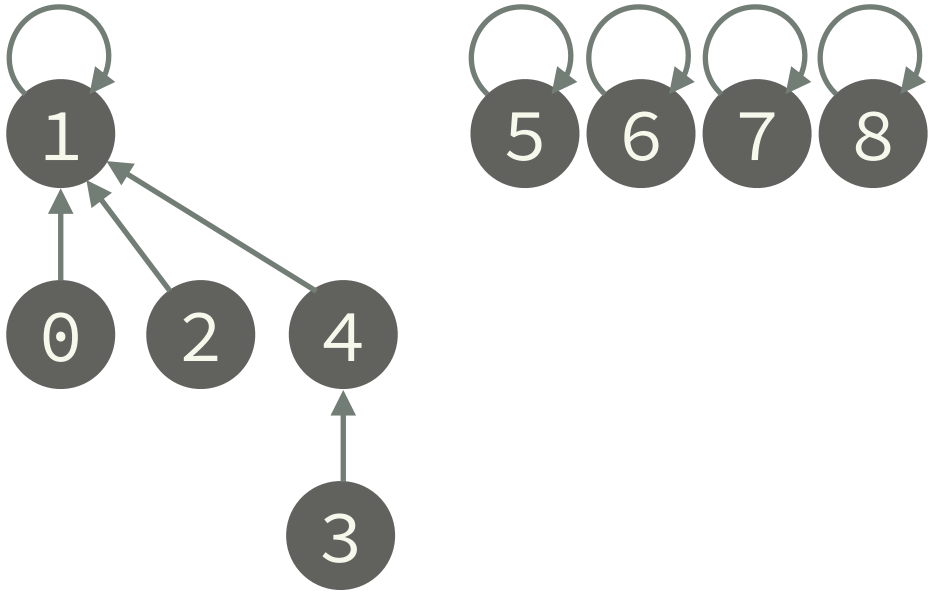

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Rationale. Reduce the likelihood of long chains.

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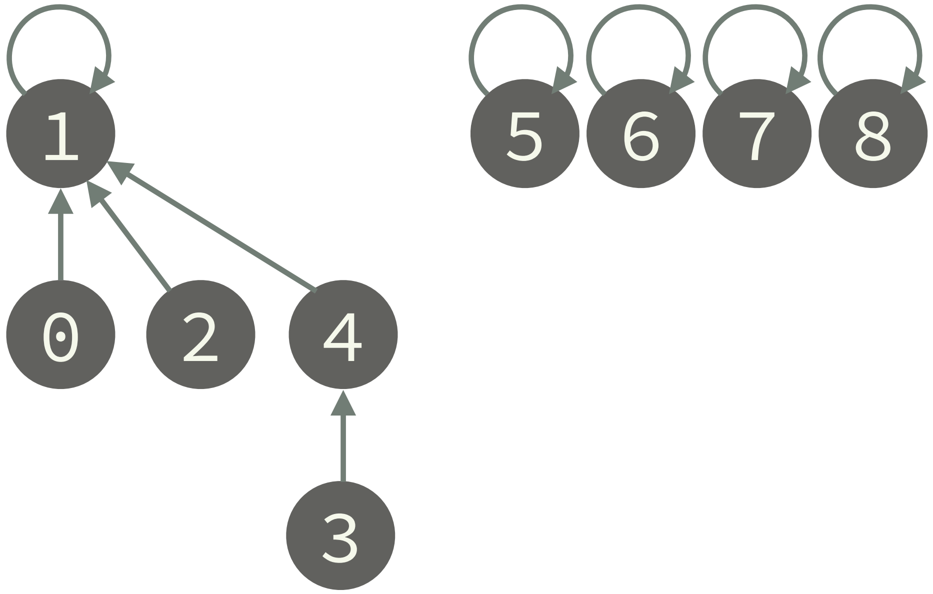
parent	1	1	1	4	**1**	5	6	7	8
size	1	**5**	1	1	2	1	1	1	1
	0	1	2	3	4	5	6	7	8



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```

Worst Case Running Time.

FIND:
UNION:

Cost Model. Number of array accesses.

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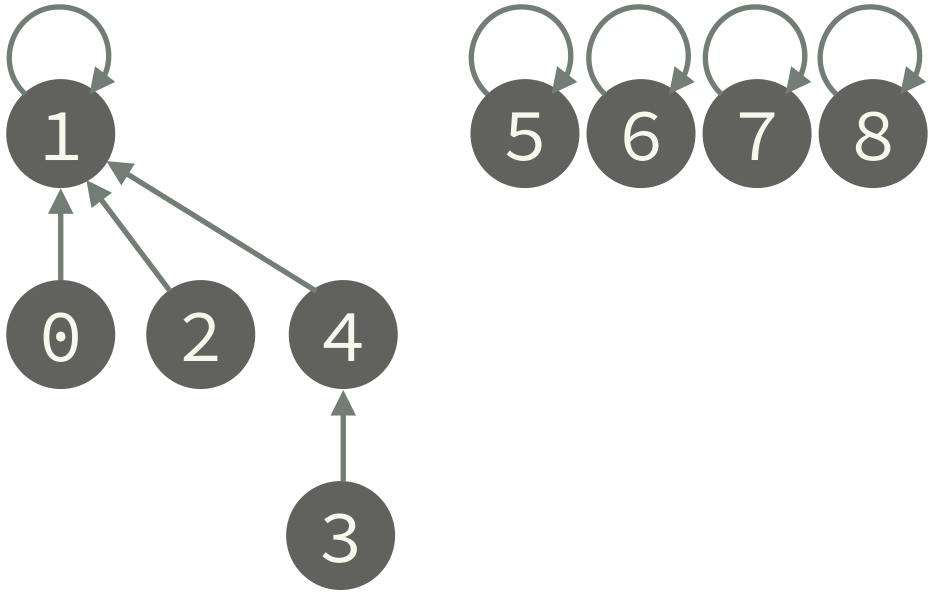
size	1	5	1	1	2	1	1	1	1
	0	1	2	3	4	5	6	7	8



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    SWAP(r1, r2)

parent[r2] = r1
size[r1]  += size[r2]
```

Worst Case Running Time.

FIND: $O(\log n)$

UNION: $O(\log n)$ 😁

Cost Model. Number of array accesses.

Weighted Quick-Union (by size)**:** Running Time

Proposition. The depth of any node in a tree of size $\boldsymbol{K}$ built using a sequence of weighted quick-union operations (by size) is $\leq \log_2 K$.

Weighted Quick-Union (by size): Running Time

Proposition. The depth of any node in a tree of size ***K*** built using a sequence of weighted quick-union operations (by size) is $\leq \log_2 K$.

Proof By Induction.

Base Case. A tree of size $K = 1$ has one node at depth $0 = \log_2 1 = \log_2 K$.

Weighted Quick-Union (by size): Running Time

Proposition. The depth of any node in a tree of size $\boldsymbol{K}$ built using a sequence of weighted quick-union operations (by size) is $\leq \log_2 K$.

Proof By Induction.

Base Case. A tree of size $K = 1$ has one node at depth $0 = \log_2 1 = \log_2 K$.

Induction Step. Let the proposition be true for every tree of size $i < K$.
Consider two trees of sizes $M > 0$ and $N > 0$, where $M \leq N$ and $N + M = K$.

Weighted Quick-Union (by size): Running Time

Proposition. The depth of any node in a tree of size $\boldsymbol{K}$ built using a sequence of weighted quick-union operations (by size) is $\leq \log_2 K$.

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Consider two trees of sizes $M > 0$ and $N > 0$, where $M \leq N$ and $N + M = K$.

By the induction hypothesis, the maximum depth in the *smaller* tree is $\leq \log_2 M$
and the maximum depth in the *larger* tree is $\leq \log_2 N$.

Weighted Quick-Union (by size): Running Time

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- The **`UNION`** operation does not affect the depths of the N nodes in the larger subtree.

Weighted Quick-Union (by size): Running Time

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Proof By Induction.

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Induction Step. Let the proposition be true for every tree of size $i < K$.
Consider two trees of sizes $M > 0$ and $N > 0$, where $M \leq N$ and $N + M = K$.

By the induction hypothesis, the maximum depth in the *smaller* tree is $\leq \log_2 M$
and the maximum depth in the *larger* tree is $\leq \log_2 N$.

- The **`UNION`** operation does not affect the depths of the N nodes in the larger subtree.
- The **`UNION`** operation increases the depth of the M nodes in the smaller subtree by 1.

Weighted Quick-Union (by size): Running Time

Proposition. The depth of any node in a tree of size $\boldsymbol{K}$ built using a sequence of weighted quick-union operations (by size) is $\leq \log_2 K$.

Proof By Induction.

Base Case. A tree of size $K = 1$ has one node at depth $0 = \log_2 1 = \log_2 K$.

Induction Step. Let the proposition be true for every tree of size $i < K$.
Consider two trees of sizes $M > 0$ and $N > 0$, where $M \leq N$ and $N + M = K$.

By the induction hypothesis, the maximum depth in the *smaller* tree is $\leq \log_2 M$
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Weighted Quick-Union (by size): Running Time

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Improvement Attempt 2.2: Weighted Quick-Union (by height)

Idea. Attach the shorter tree to the longer tree.

Rationale. Isn't the height of the tree what we want to optimize?

Modification. Add an array to record the height of the tree rooted at each element.

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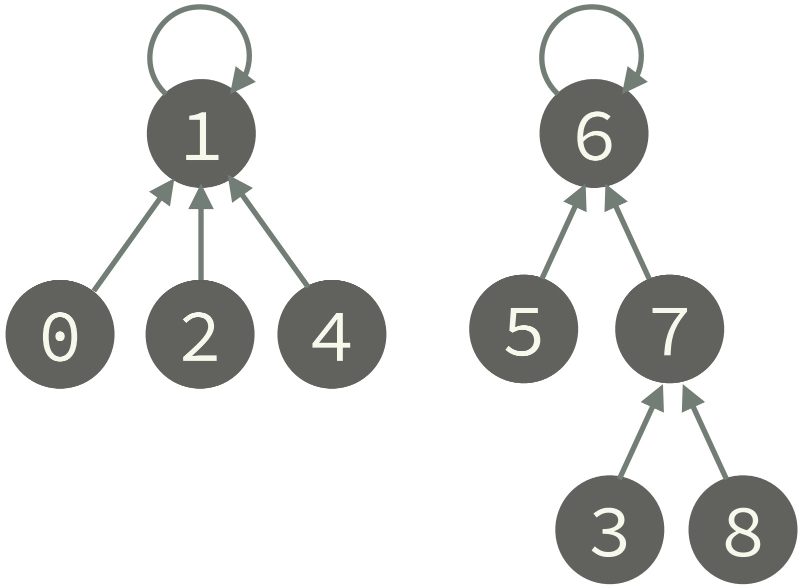
height	0	1	0	0	0	0	2	1	0



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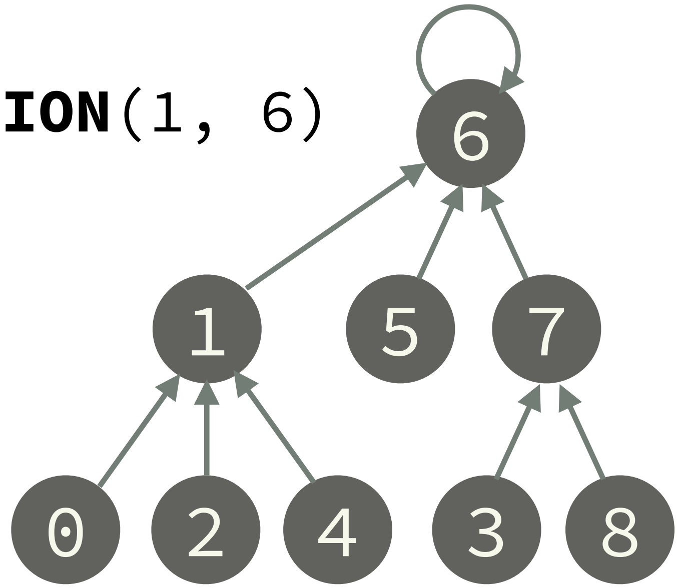

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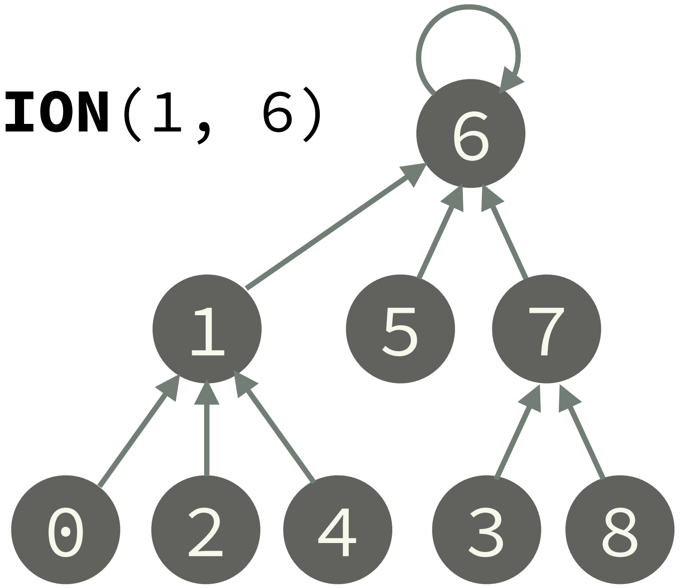

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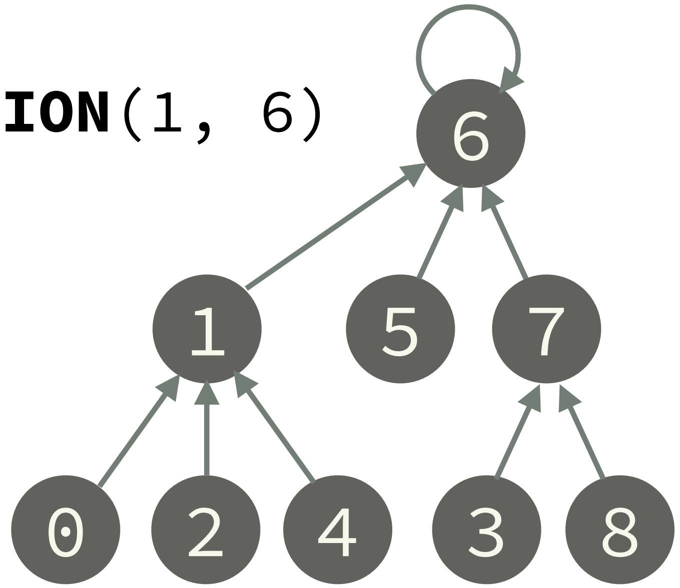

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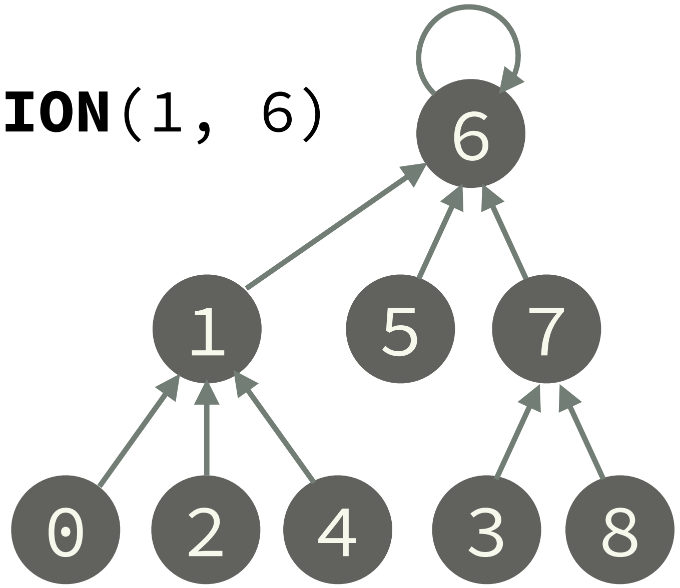

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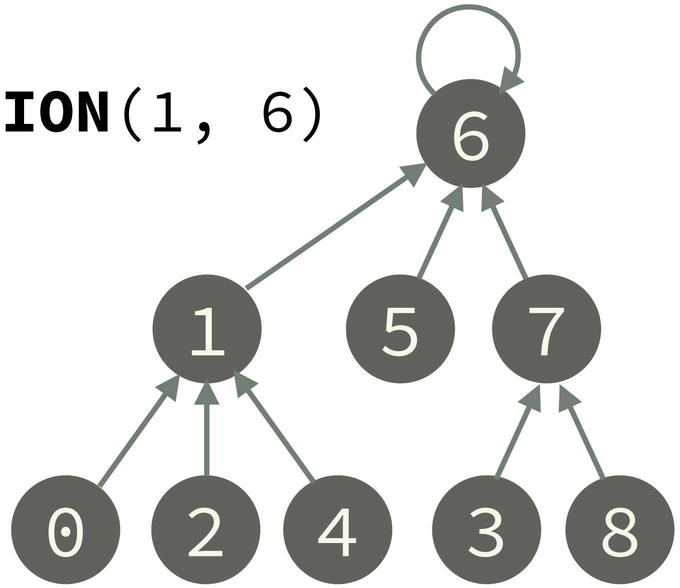

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Important. Height changes **only** when two trees of the same height are merged.

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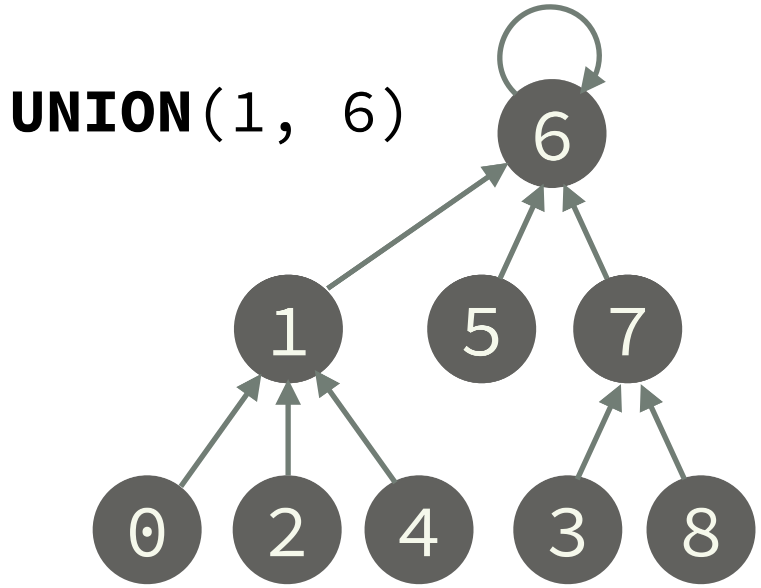

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Worst Case Running Time.

FIND: $O(\log n)$
UNION: $O(\log n)$

Cost Model. Number of array accesses.

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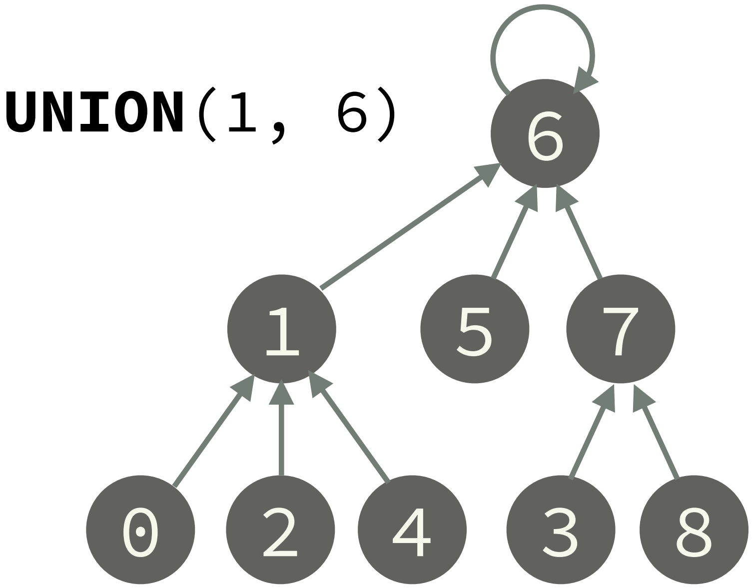

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I see a proof coming ...

Weighted Quick-Union (by height): Running Time

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Proof. From the proof of proposition 1, a tree of height $h > \log_2 N$ has more than $2^{\log_2 N} > N$ nodes. This is a contradiction, as there are only N nodes in the tree! ■

Quiz # 3

Draw a tree that can be the result of weighted quick-union-by-size but can't be the result of weighted quick-union-by-height.

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Quiz # 3

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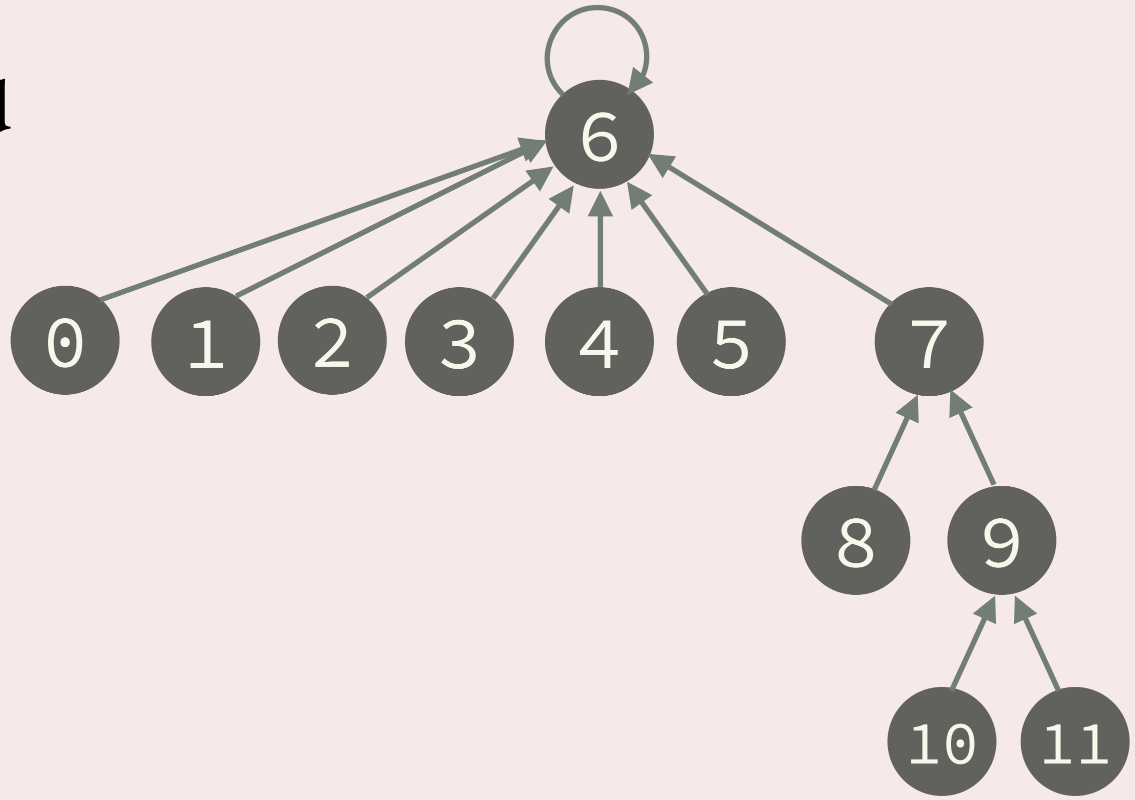

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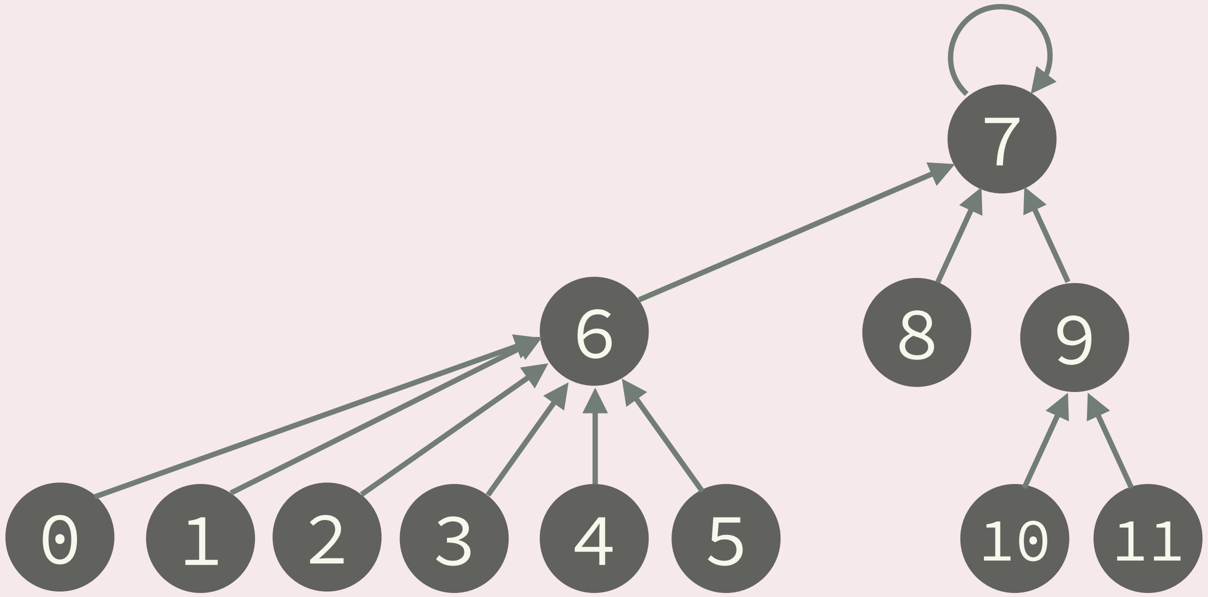

Union-Find: Running Time Summary

	Quick-**Find** (array)	Quick-**Find** (linked-list)
`FIND`	$O(1)$	$O(1)$
`UNION`	$O(n)$	$O(n)$
Sequence of n `UNION` operations:	$O(n^2)$	$O(n \log n)$

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🤔

Can we do better?

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FIND(p)

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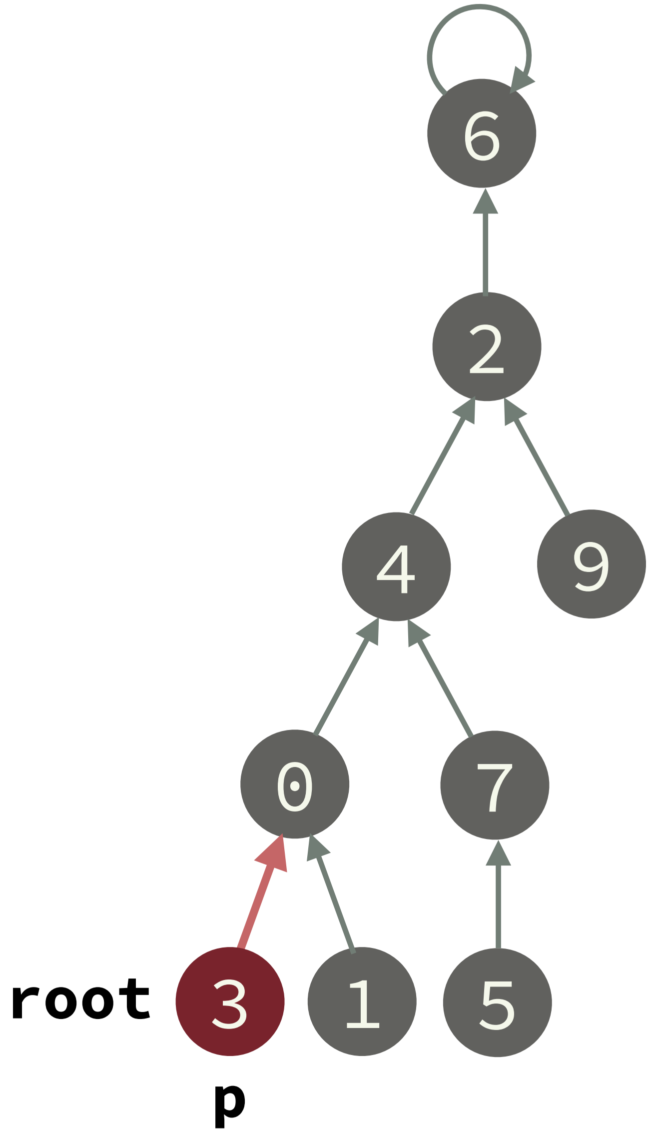
1. get the root as usual

Example. **`FIND`**`(3)`



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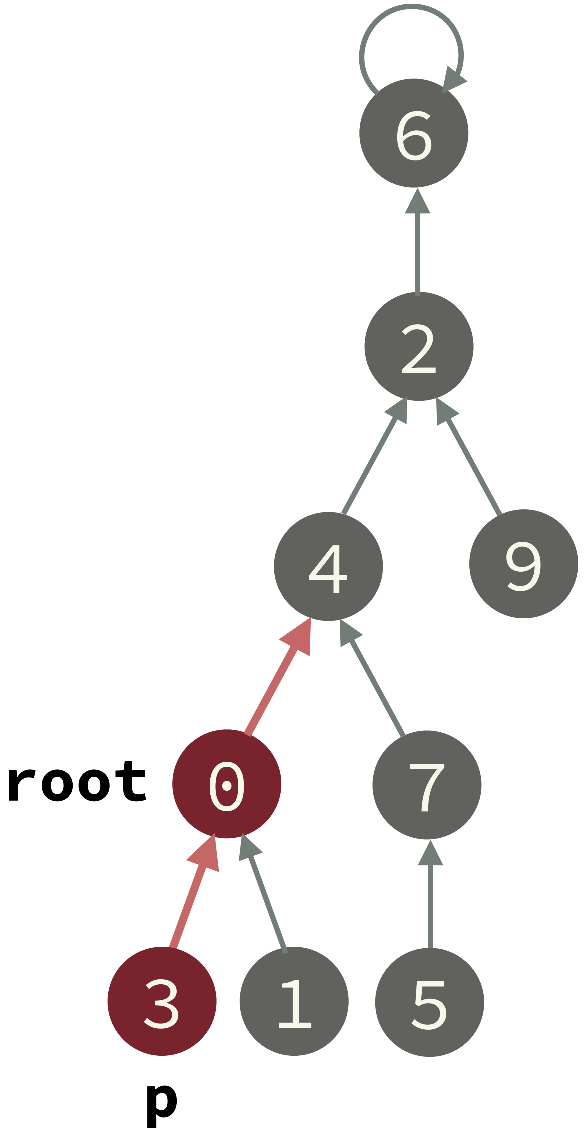

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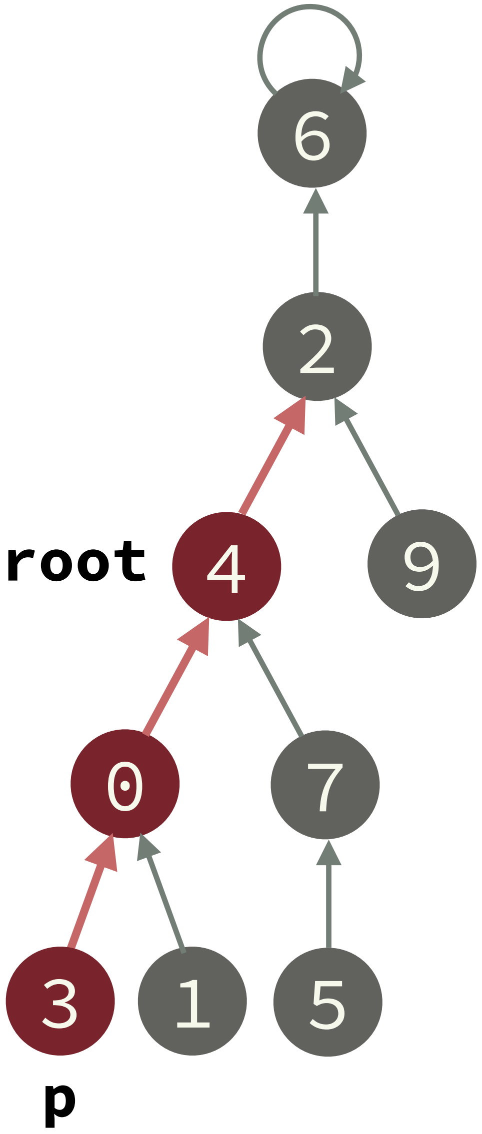

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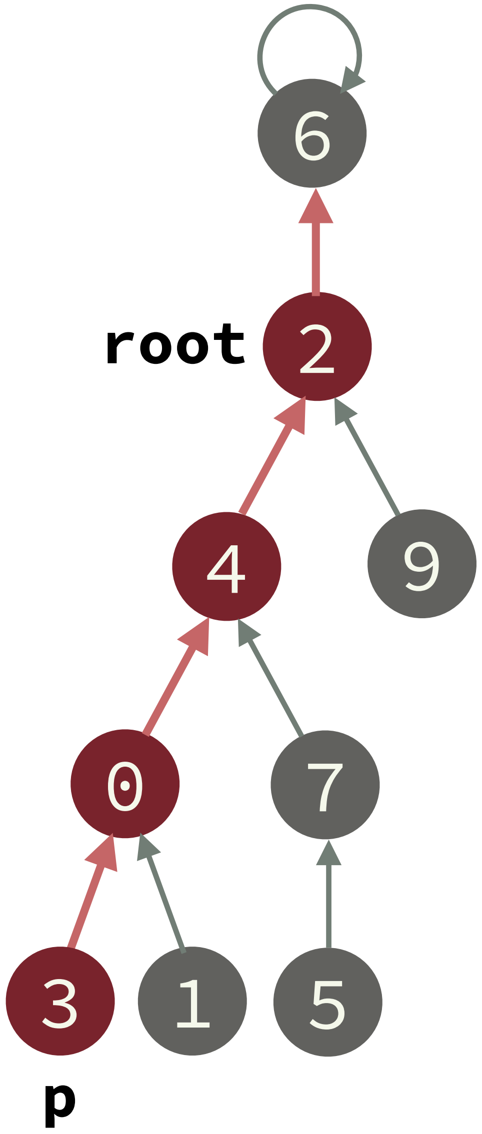

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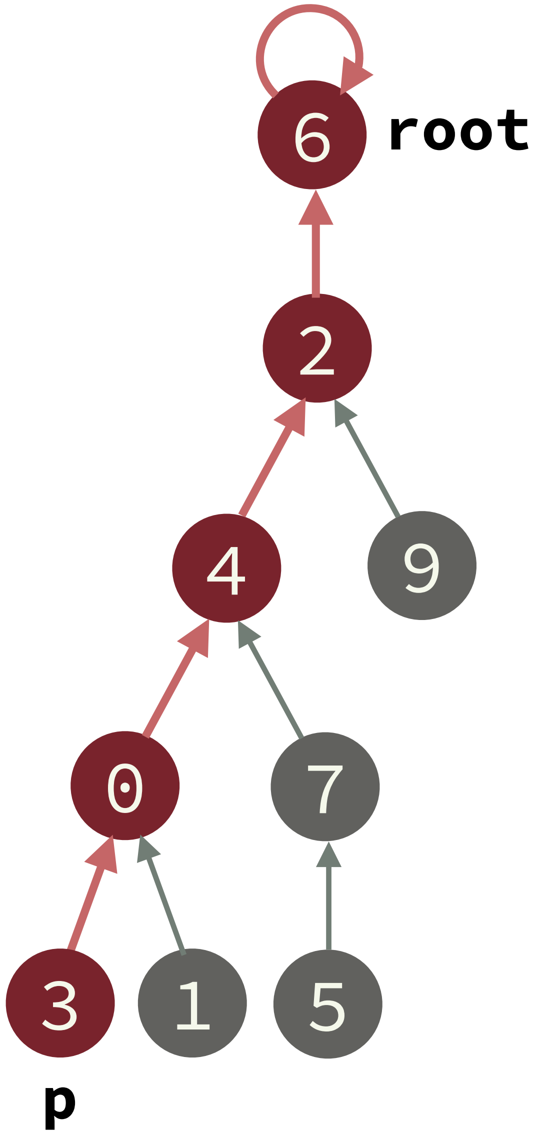

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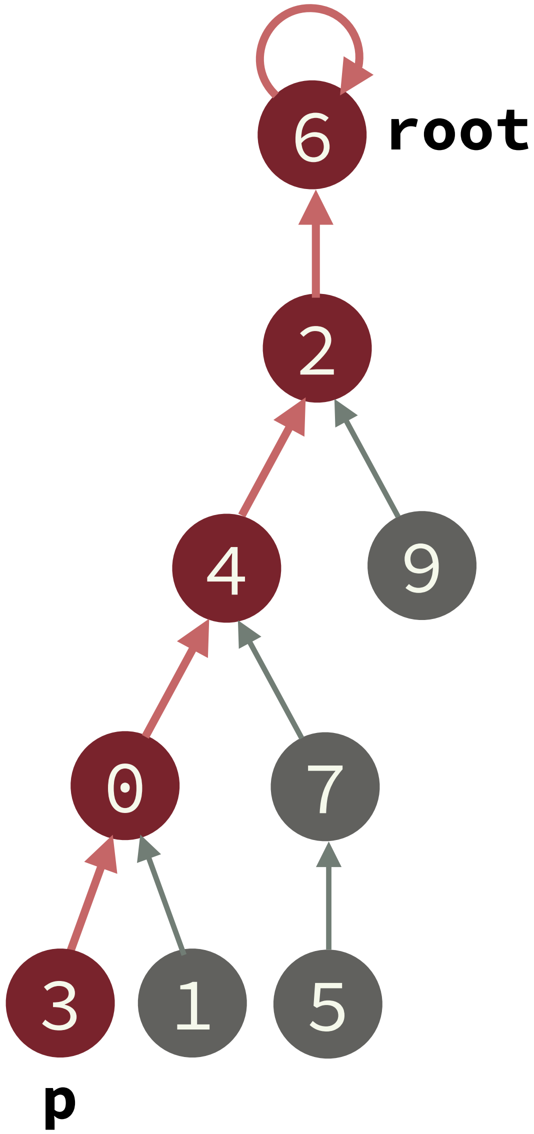

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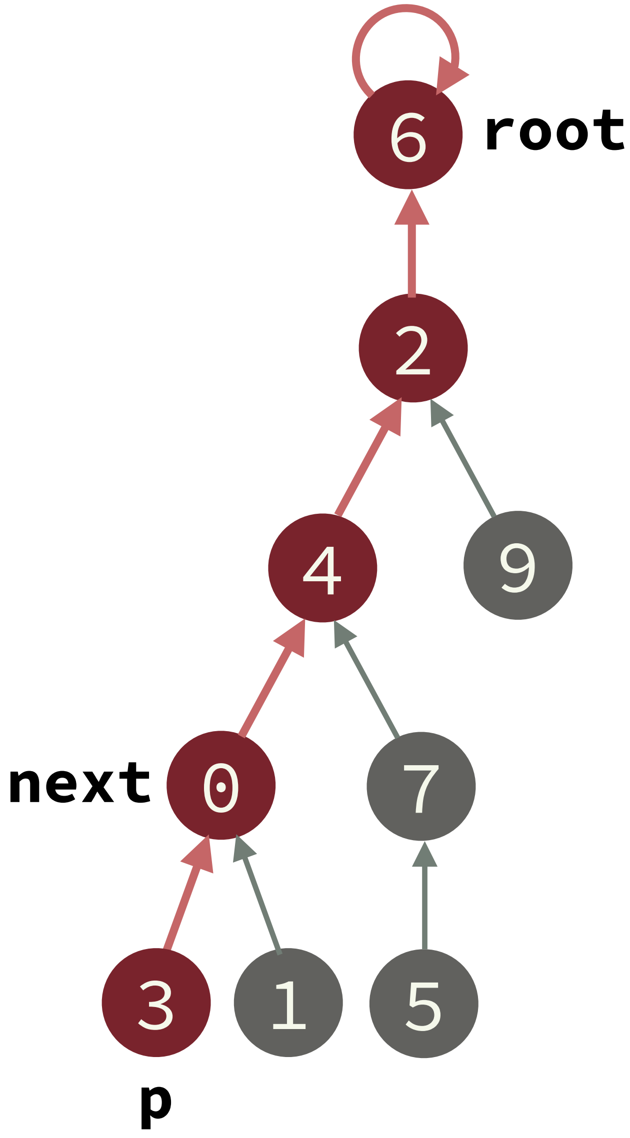

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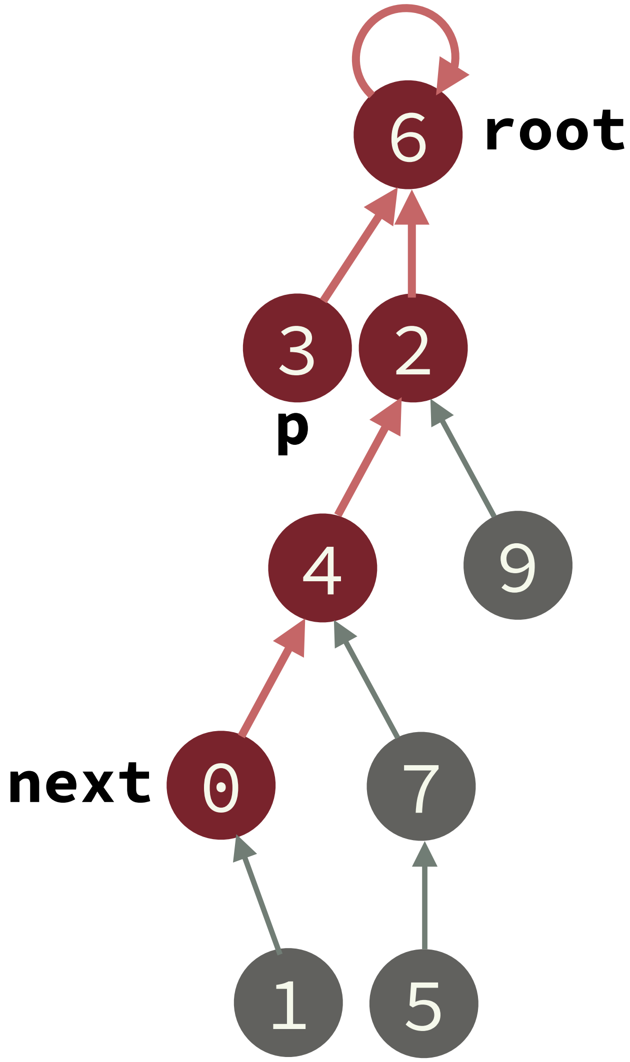

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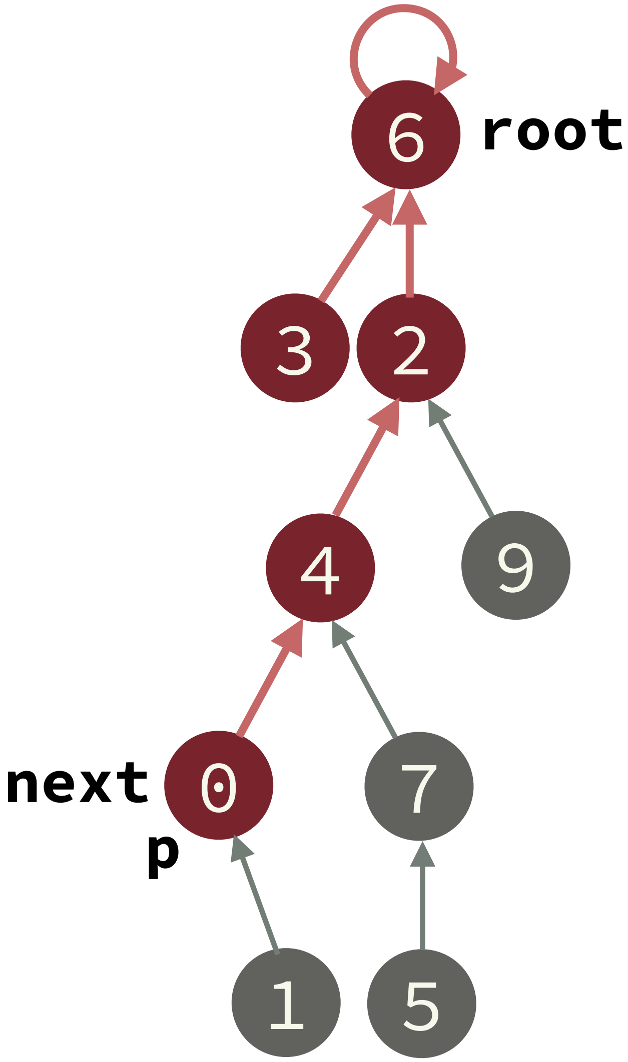

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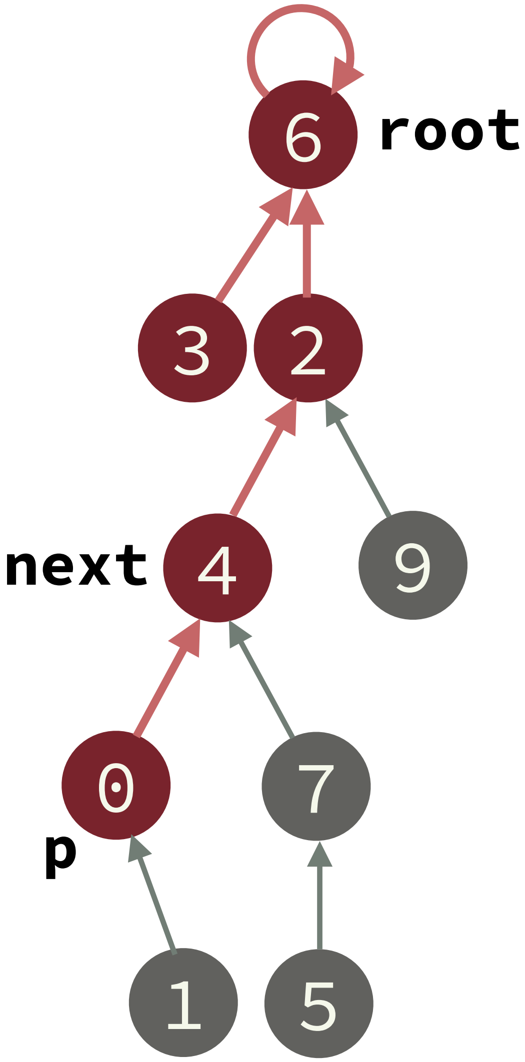

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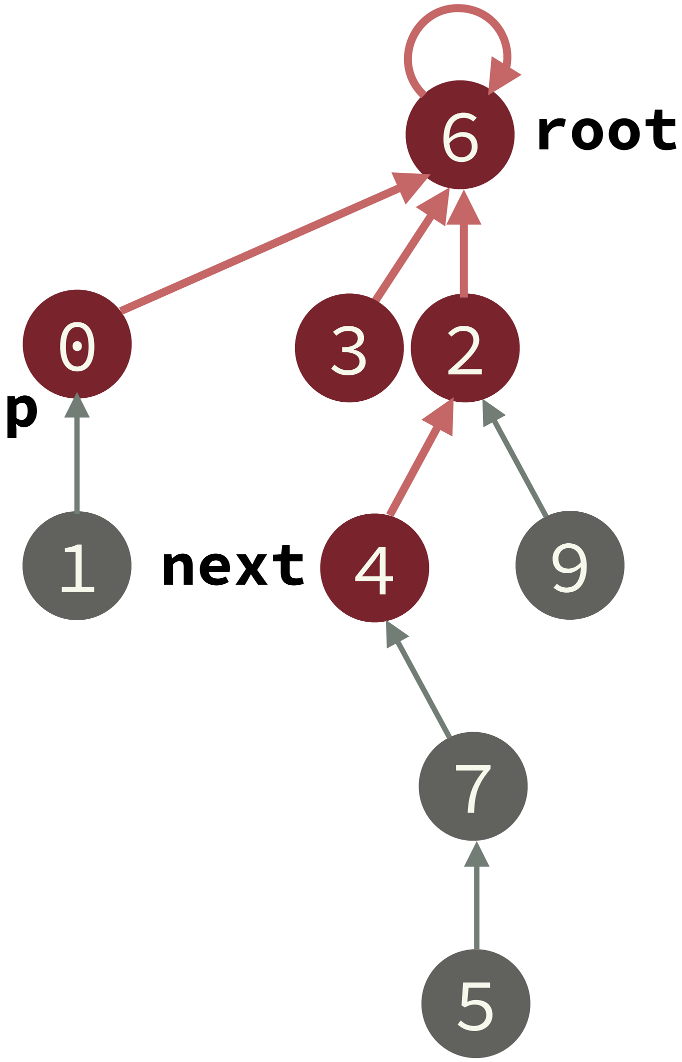

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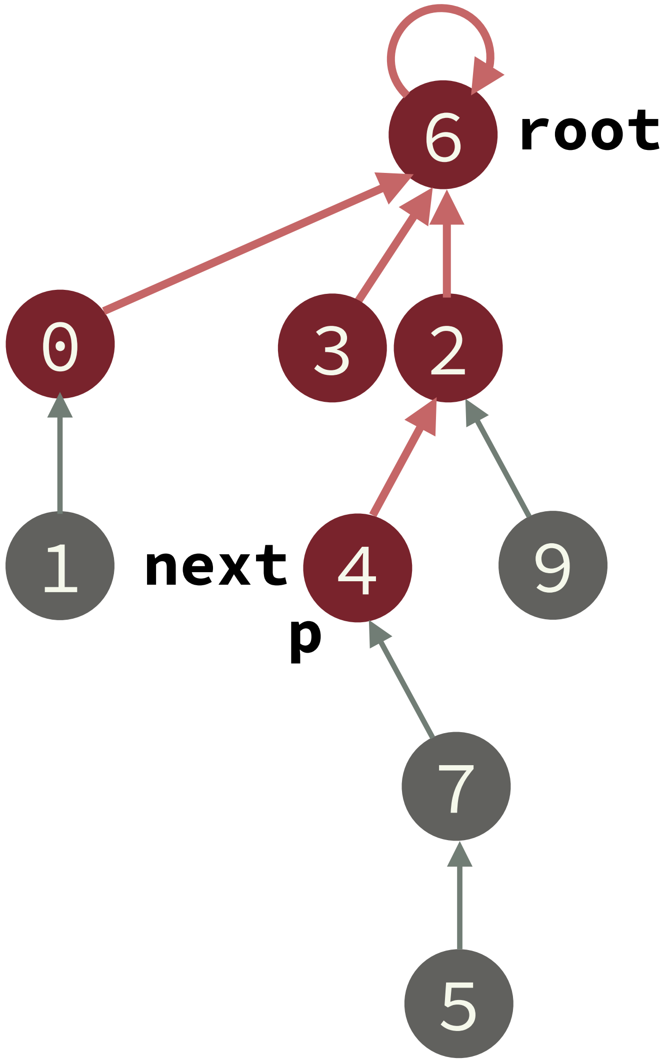

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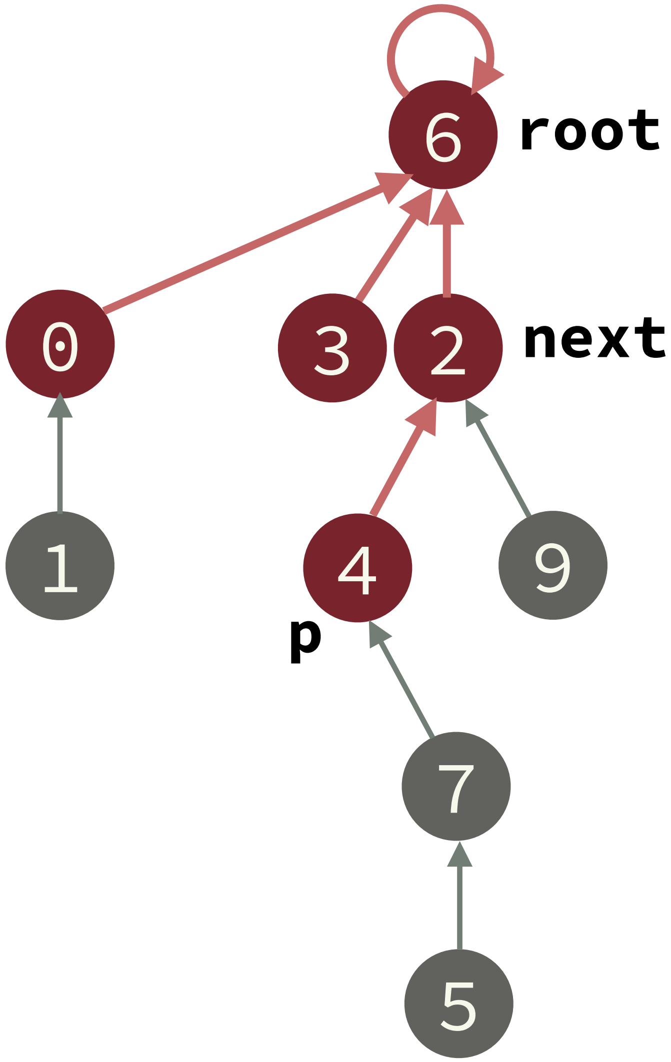

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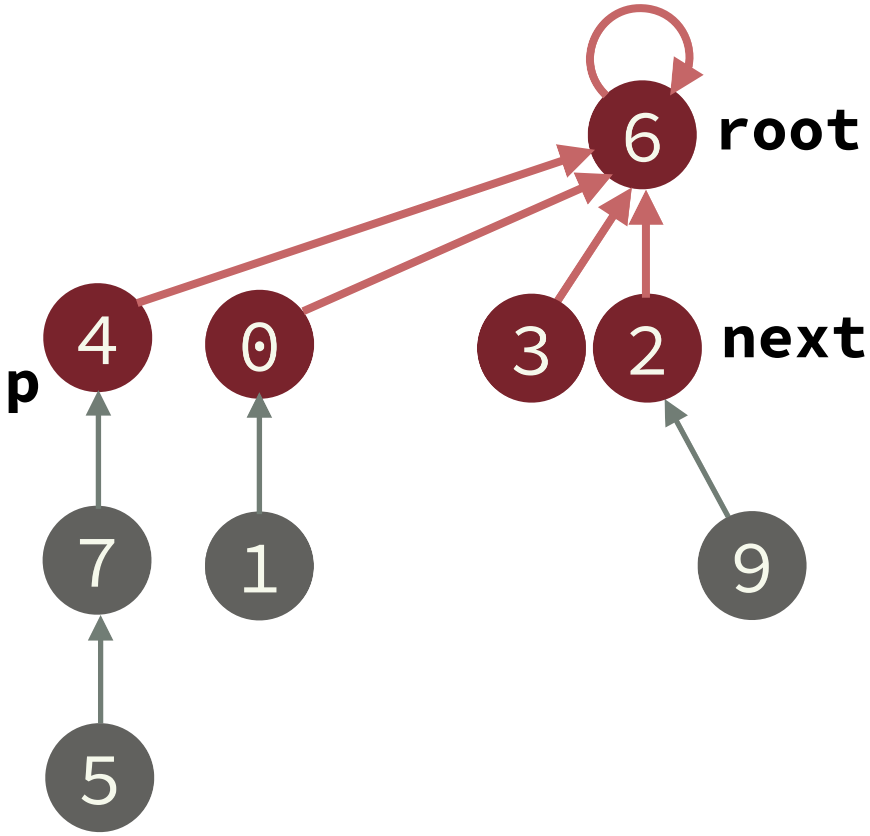

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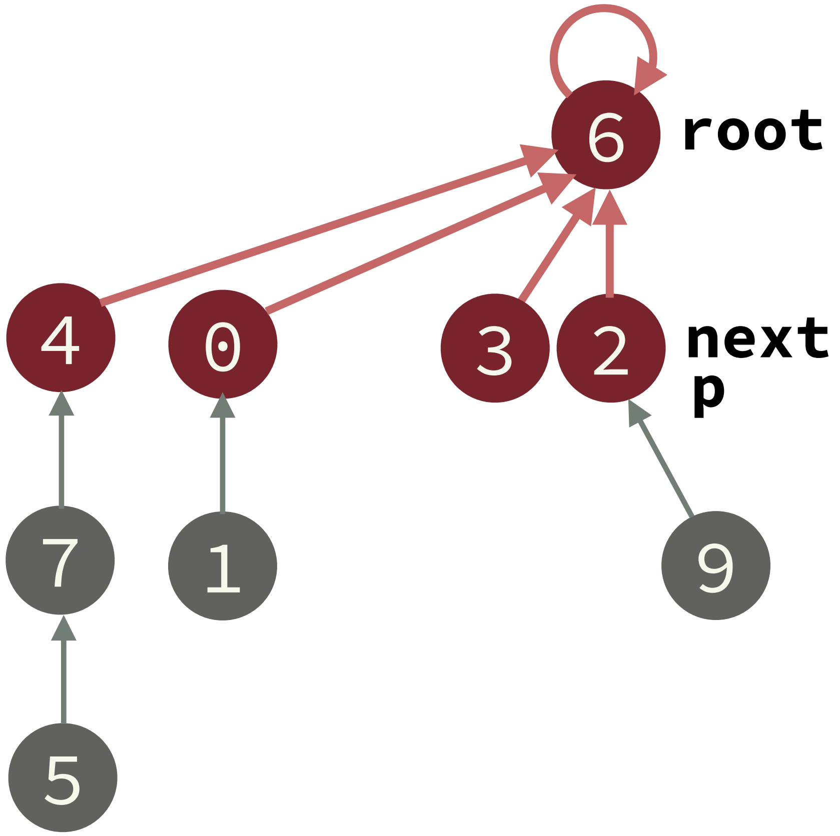

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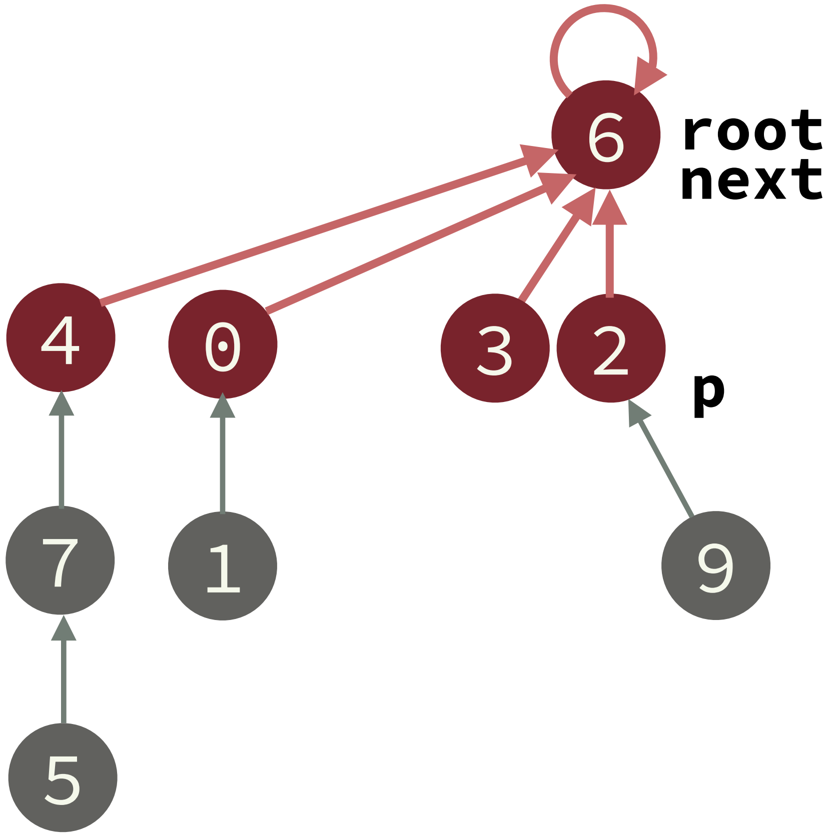

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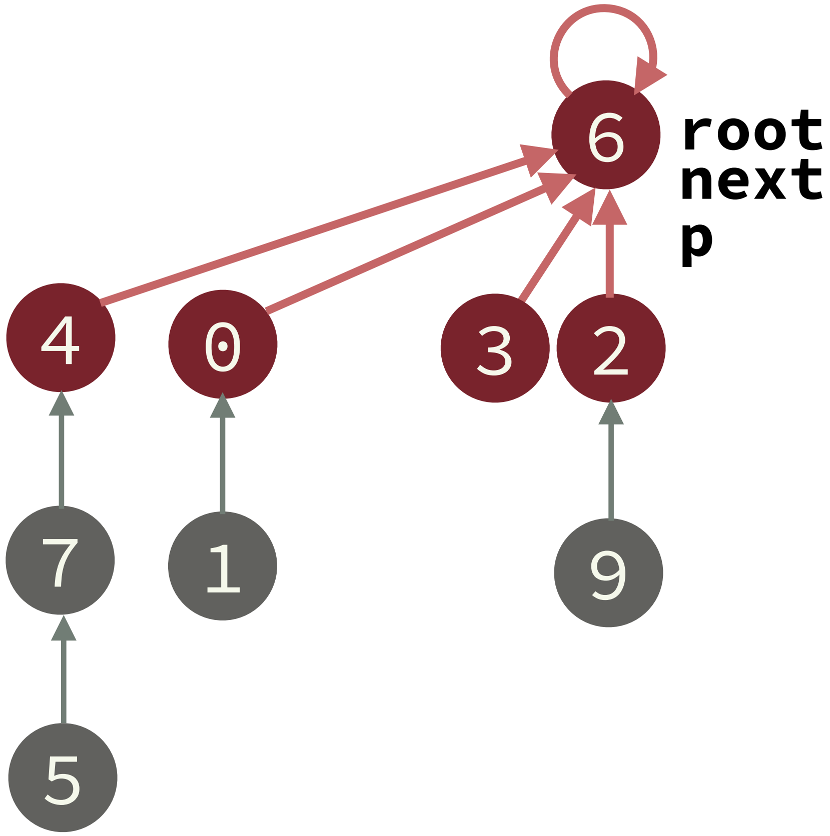

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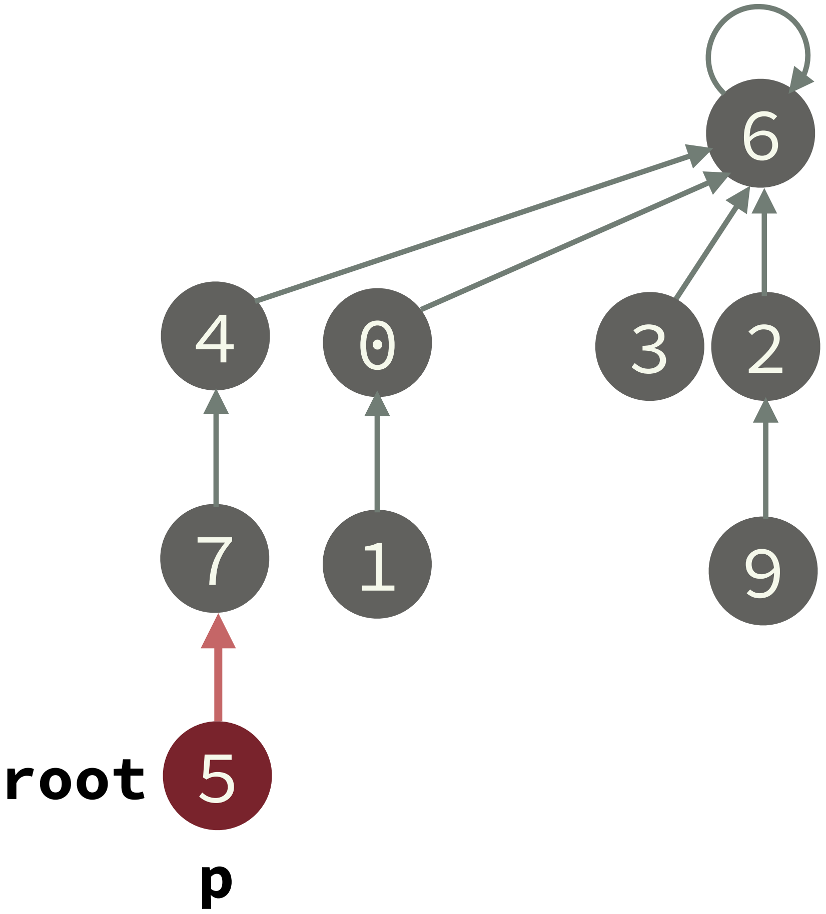

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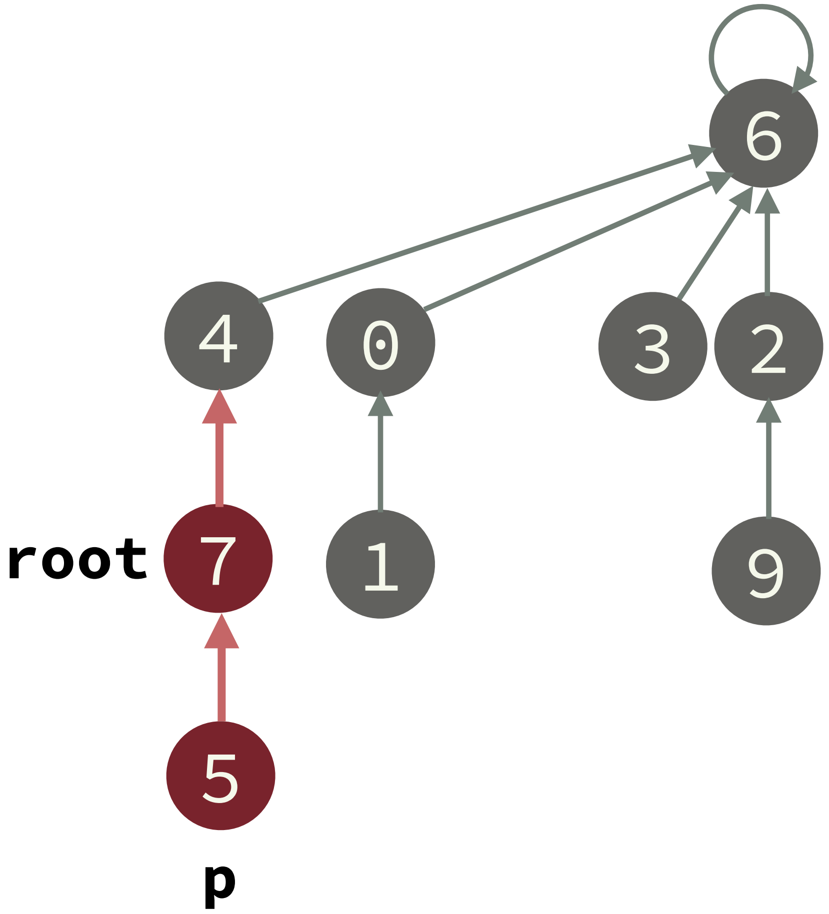

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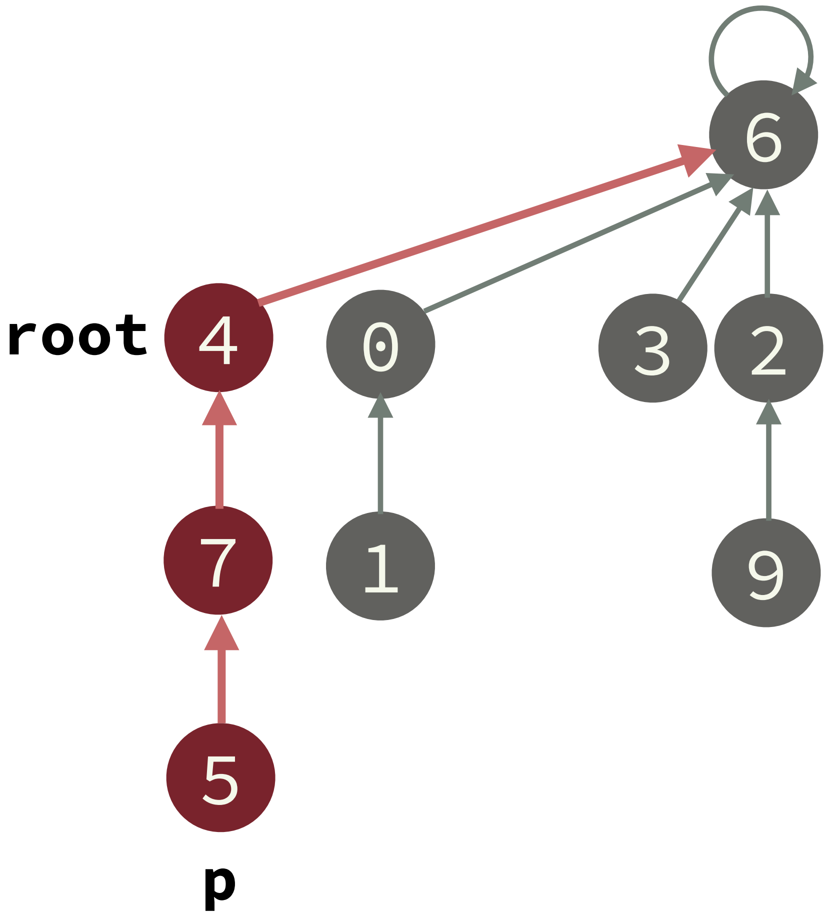

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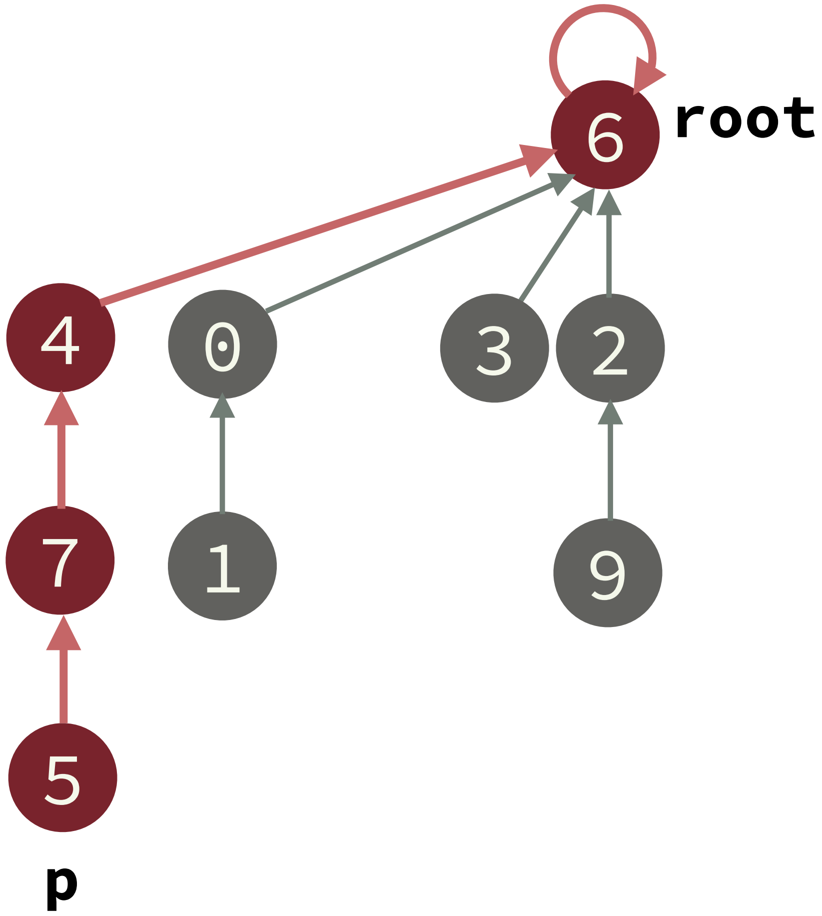

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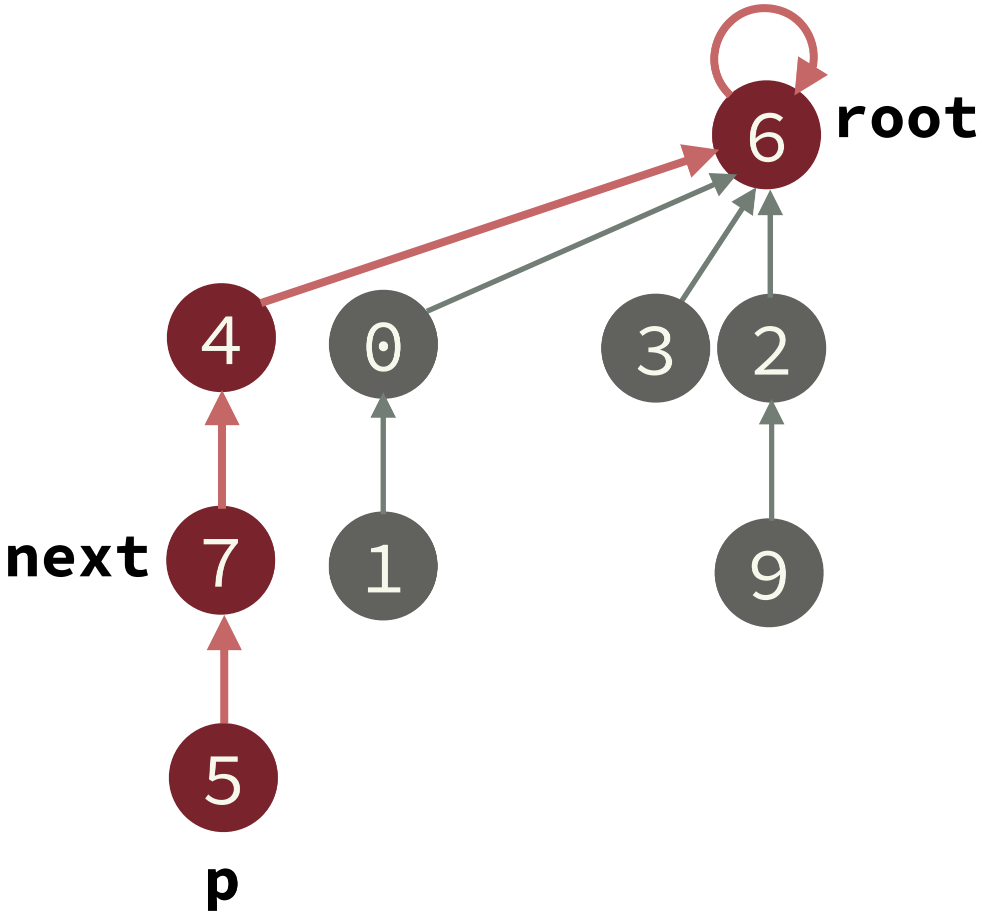

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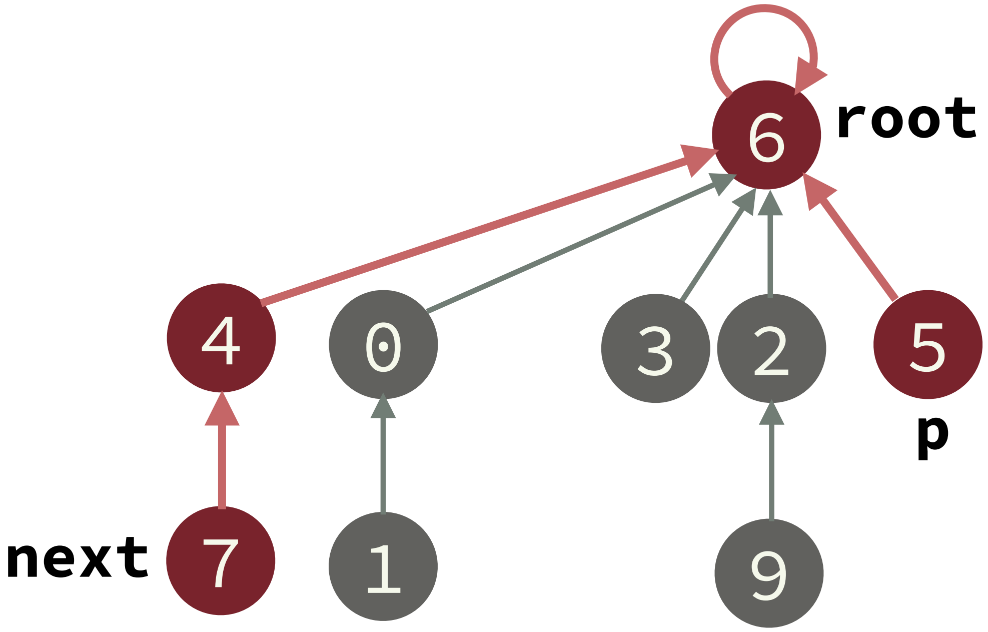

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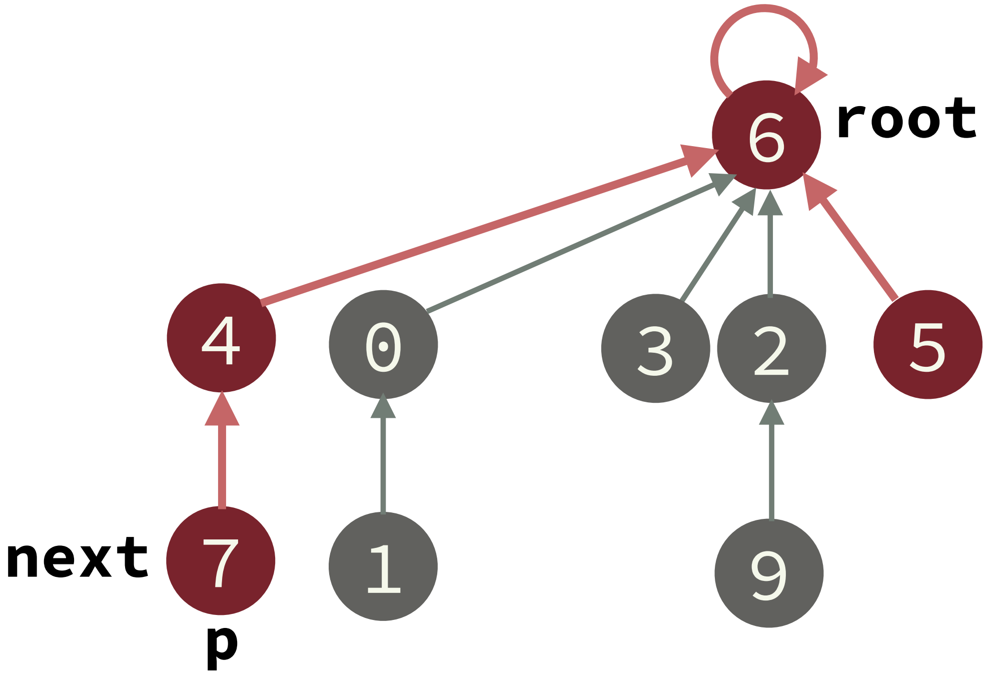

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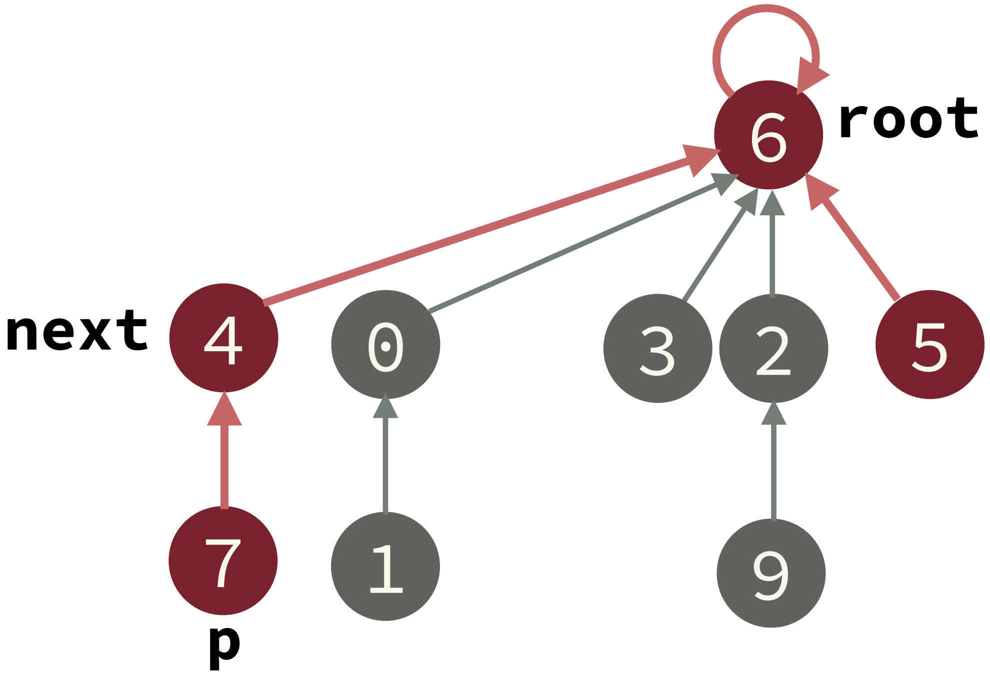

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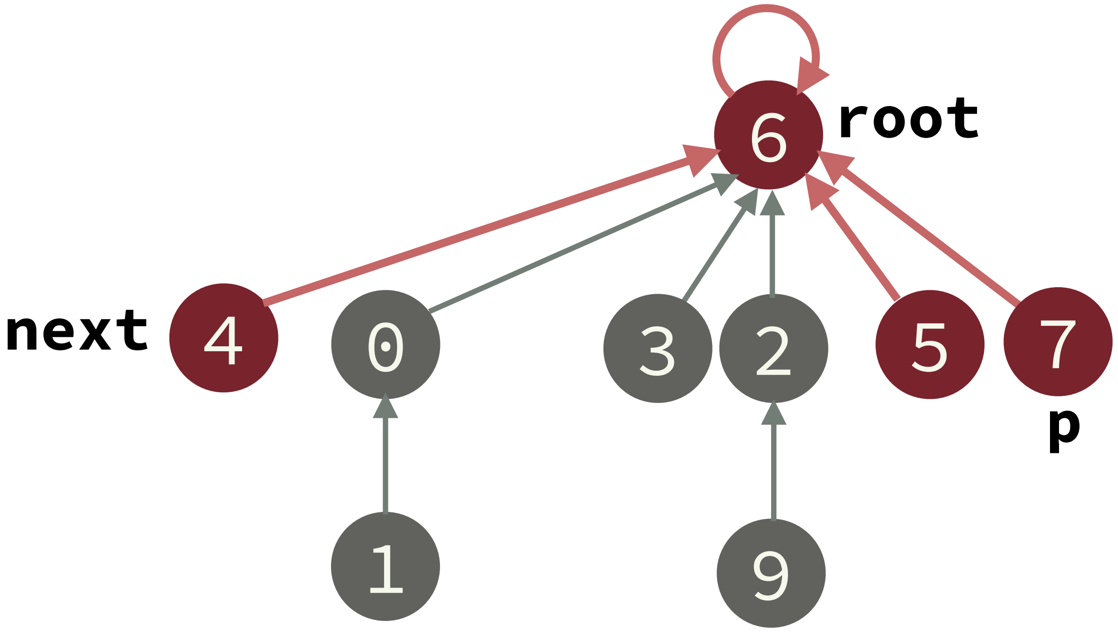

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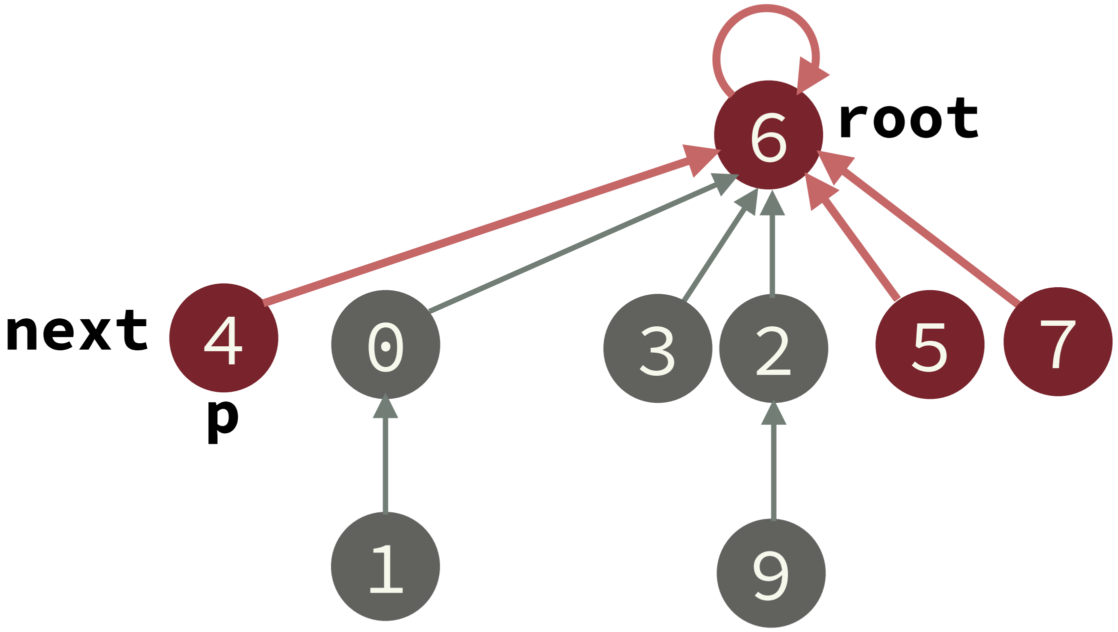

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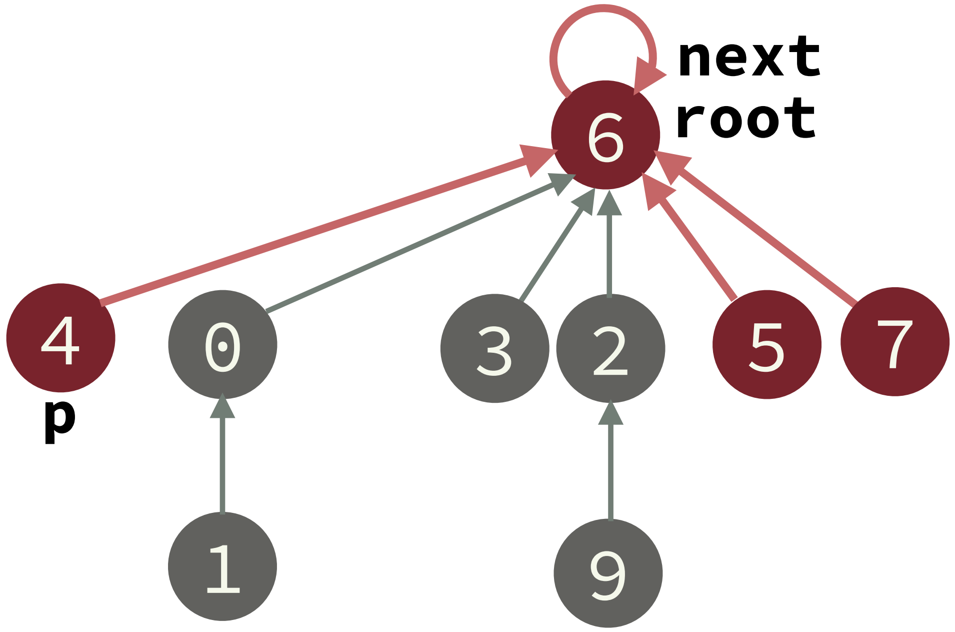

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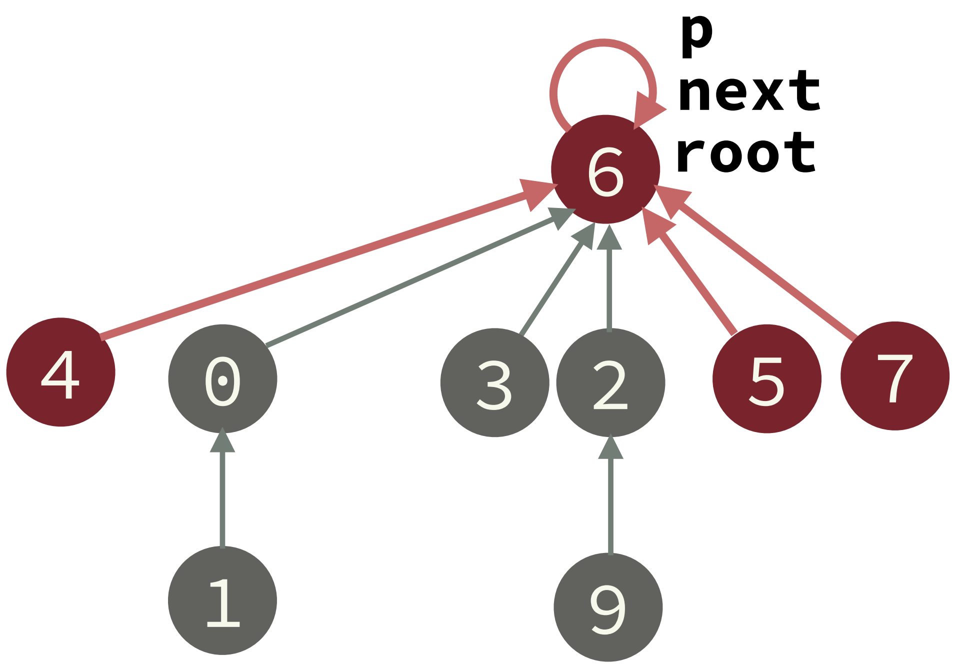

🎉 The more expensive **`FIND`** operations are performed, the flatter the tree becomes!

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Inverse Ackermann function
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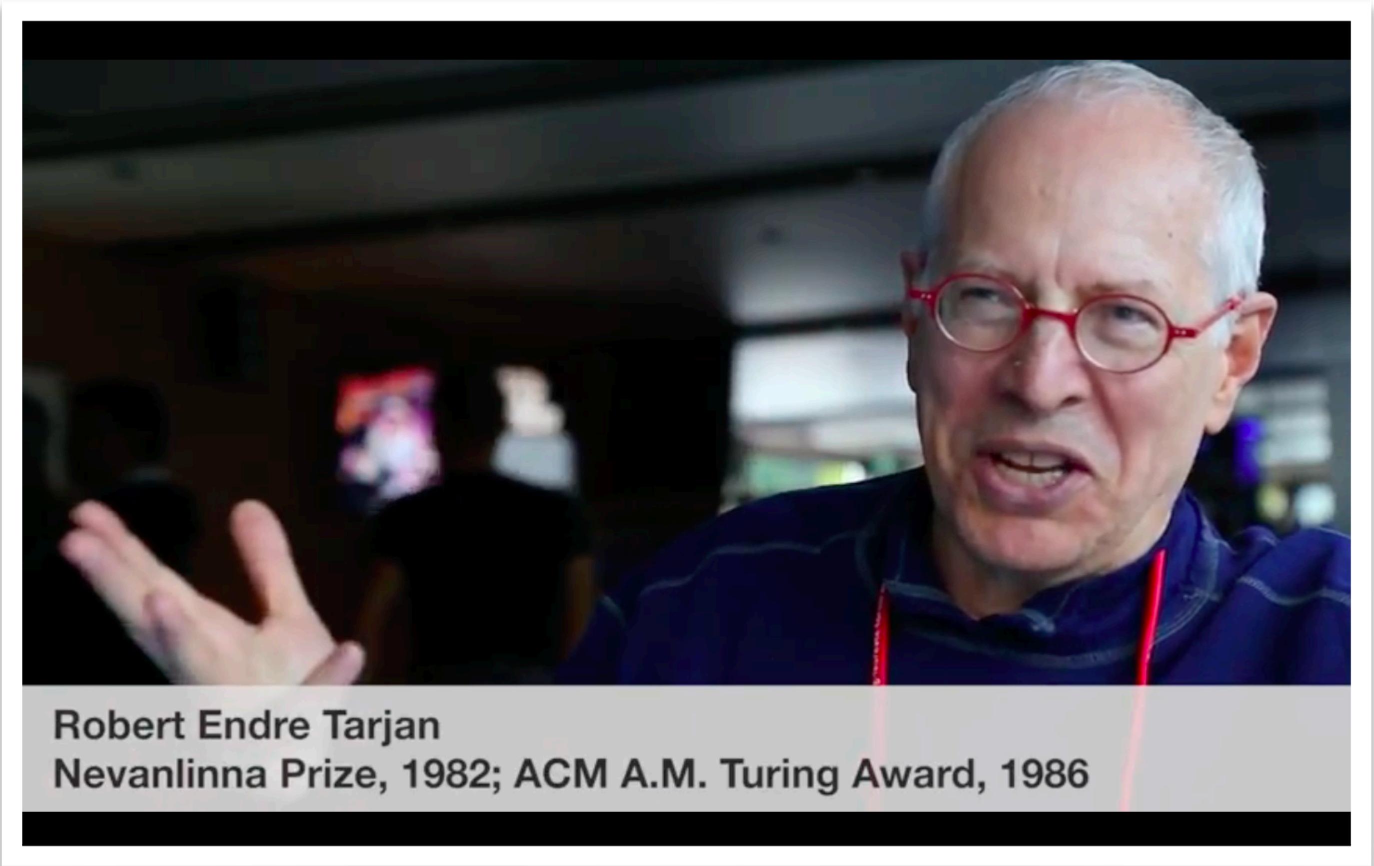

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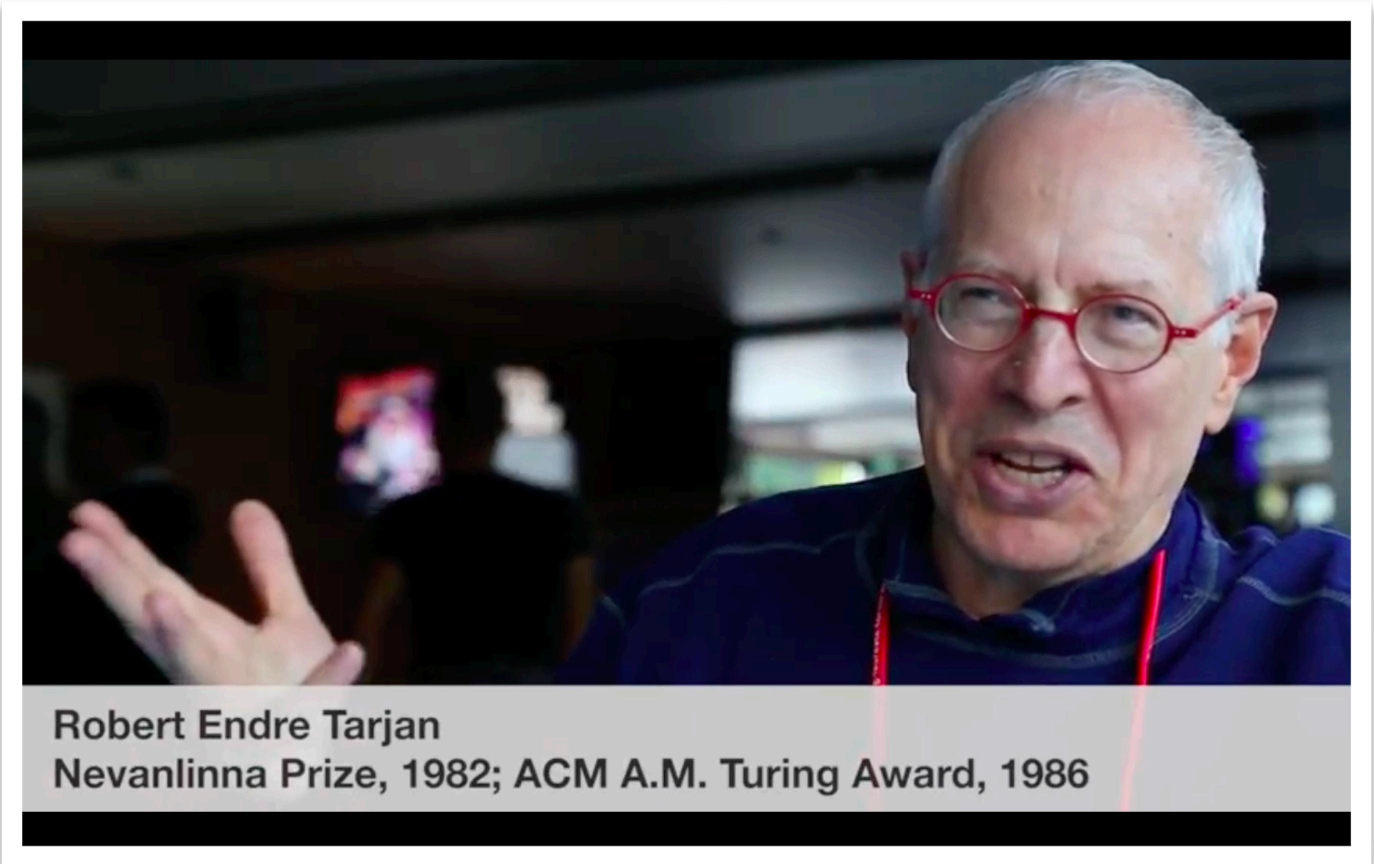

Robert Endre Tarjan
Nevanlinna Prize, 1982; ACM A.M. Turing Award, 1986

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E.g. Assign random indices to the elements and use them instead of the size in weighted quick-union (by size).

Result. Almost same performance!

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Other Optimizations.

Get rid of the `height[]` array in weighted quick-union.

How?

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