**CS**11313 - **Fall** 2021

# Design & Analysis of Algorithms

## NP Completeness

Ibrahim Albluwi

# Reductions

A *reduction* from problem $X$ to problem $Y$:
An algorithm for solving problem $X$ that includes a solver of problem $Y$ as a subroutine.

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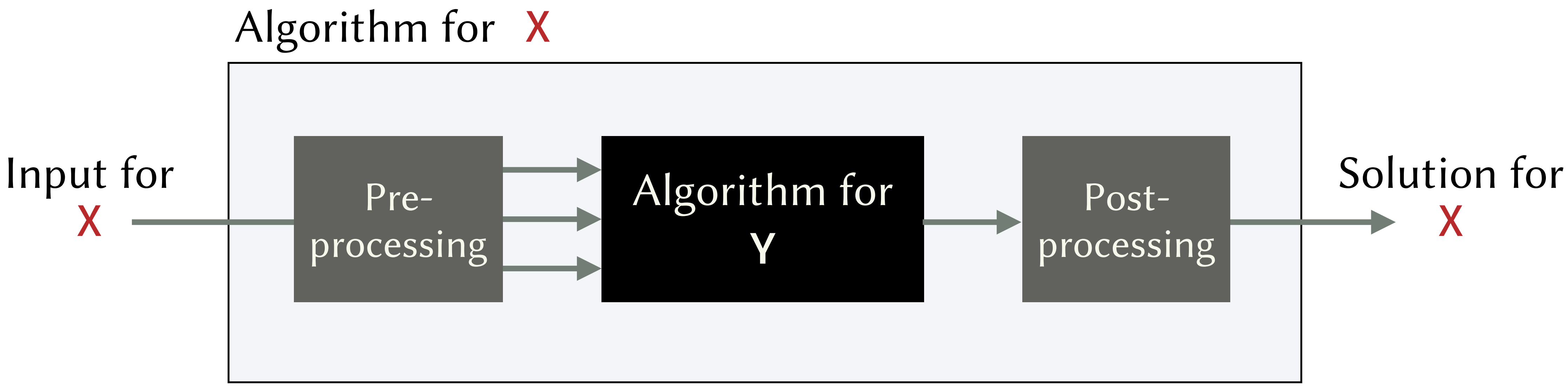


Total cost for solving $X$ = Cost of solving $Y$ + Cost of reduction

Y might be called multiple time (typically 1 call)

Typically less than the cost of solving Y

*Figure adapted from a slide by Kevin Wayne*

# Reductions

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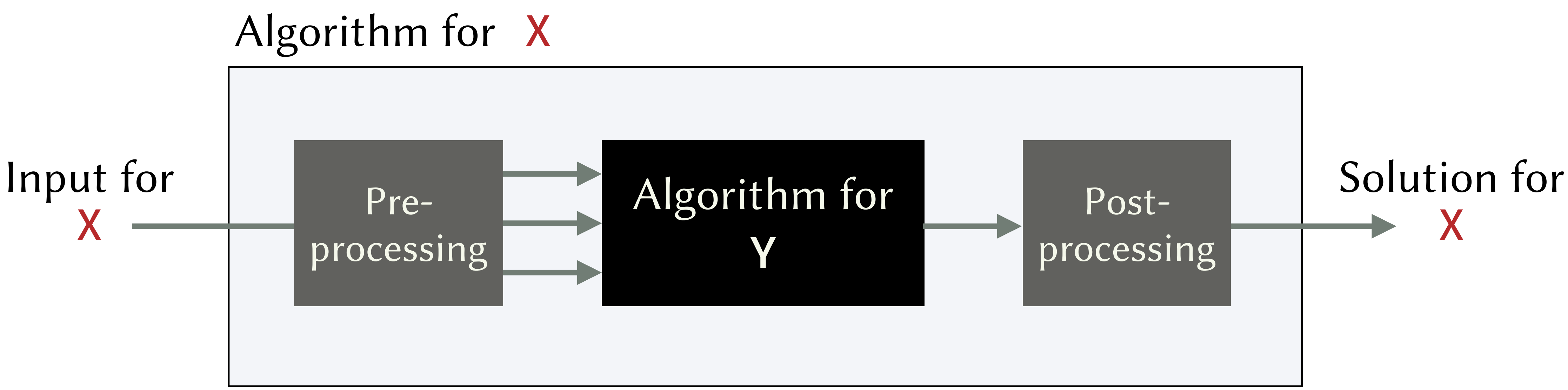
An algorithm for solving problem $X$ that includes a solver of problem $Y$ as a subroutine.

Problem $X$ *reduces* to problem Y (denoted as $X \leqslant Y$): An algorithm for solving $Y$ can be used to solve $X$.

Problem $X$ *polytime-reduces* to problem $Y$ ($X \leqslant_p Y$): An algorithm for solving $Y$ can be used to solve $X$ in addition to a polynomial-time amount of work.

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# Reductions (Examples)

`LINEAR`
Given $b$ and $c$, solve $bx + c = 0$

`QUADRATIC`
Given $a$, $b$ and $c$, solve $ax^2 + bx + c = 0$

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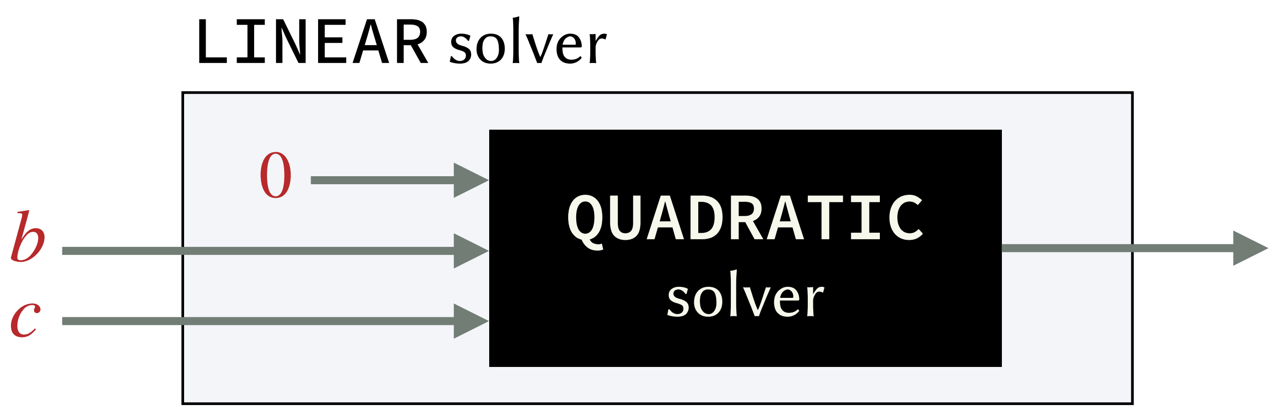

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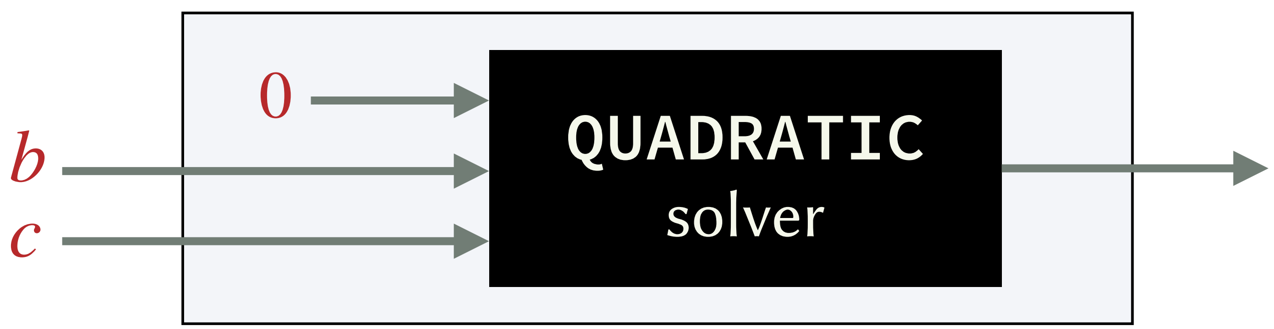
LINEAR solver




**SELECT**
Given a list of elements, find the $k^{th}$ largest element.

**SORT**
Given a list of elements, order the elements in non-decreasing order.

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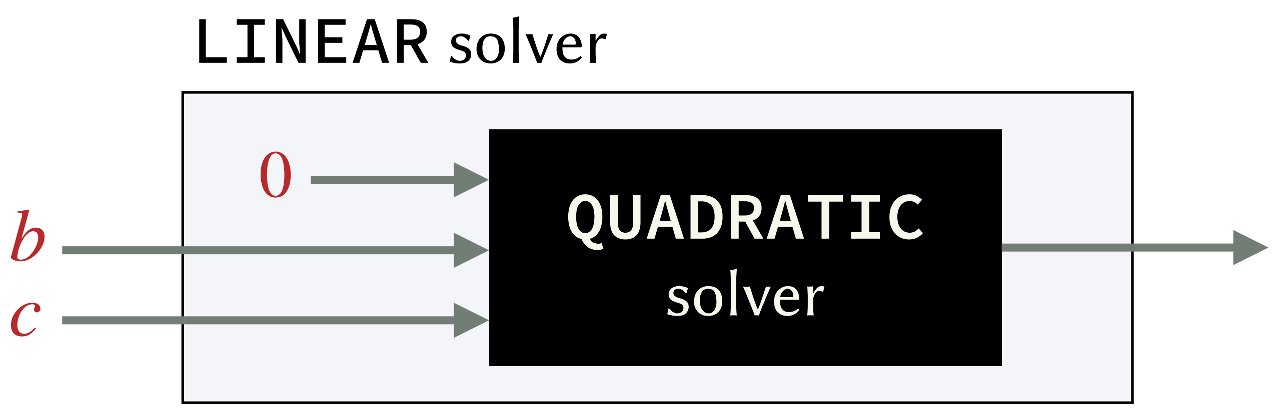


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## SELECT reduces to SORT

Use SORT to sort the elements and then report the element of rank $k$.

## SORT reduces to SELECT

Sort the elements by repeatedly using SELECT to find the next largest element.

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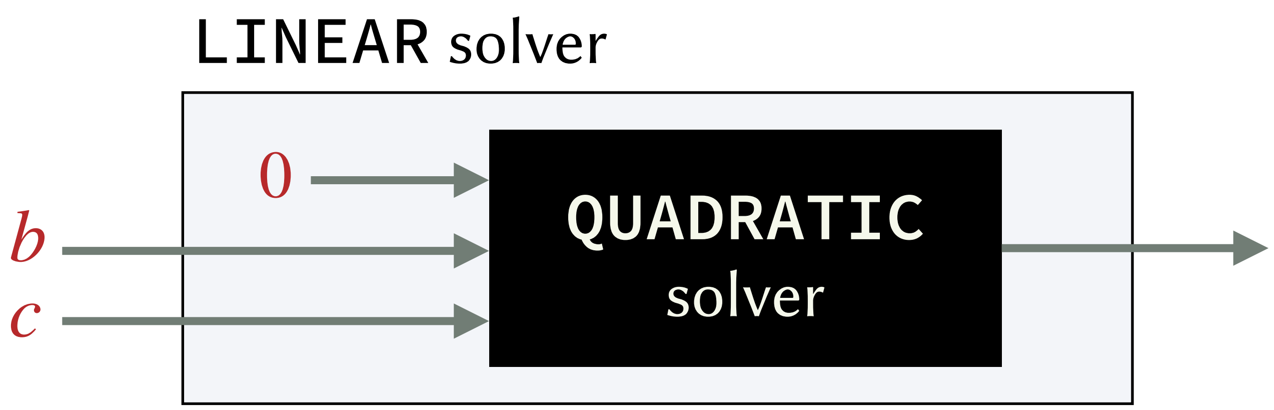


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Running Time. $\underbrace{O(N \log N)}_{\text{SORT}} + \underbrace{O(1)}_{\text{reduction}}$

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Running Time. $\underbrace{O(N)}_{\text{SELECT}} \times \underbrace{O(N)}_{\text{reduction}}$

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**SSSP** (Single Source Shortest Paths)

Given a graph $G$ and a source vertex $s$, find the shortest path from $s$ to every vertex in $G$.

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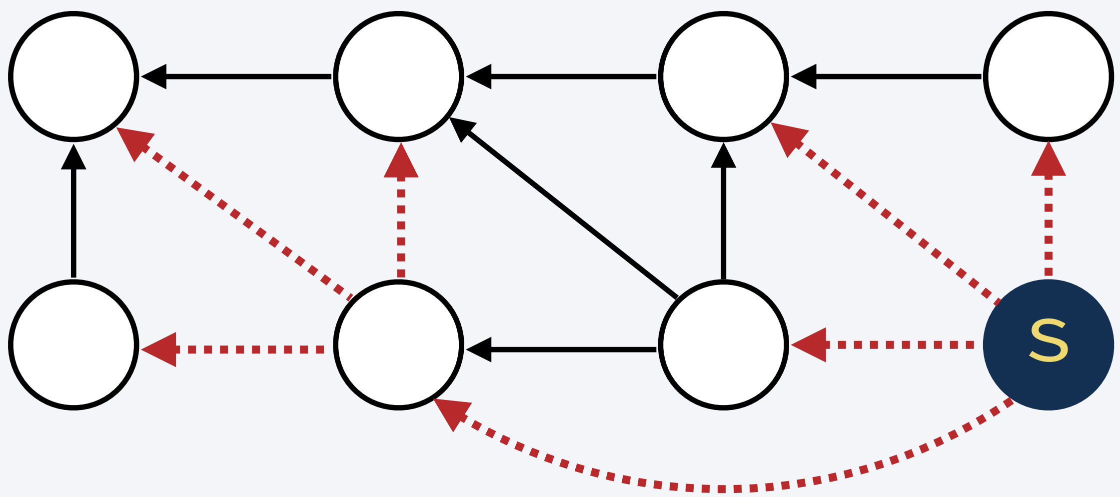


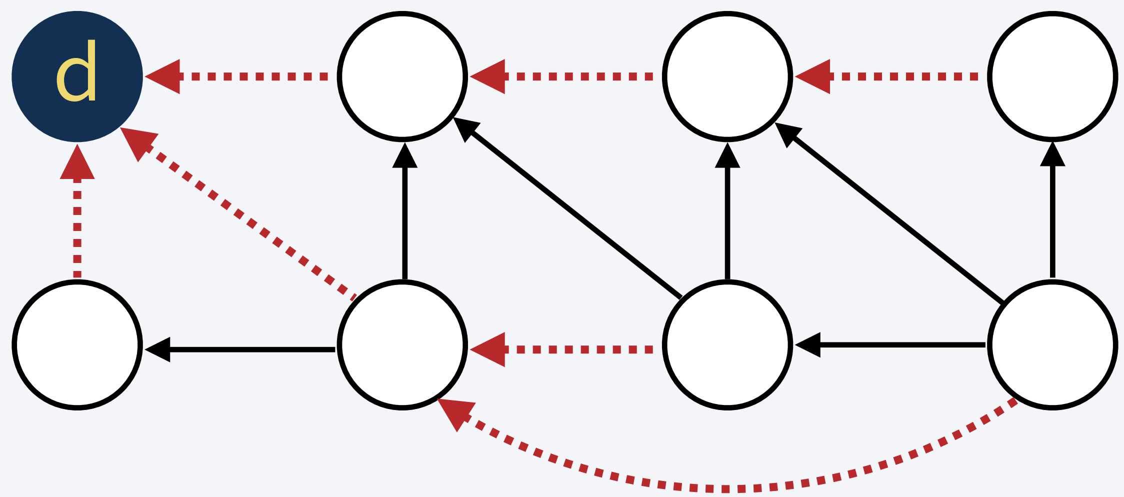

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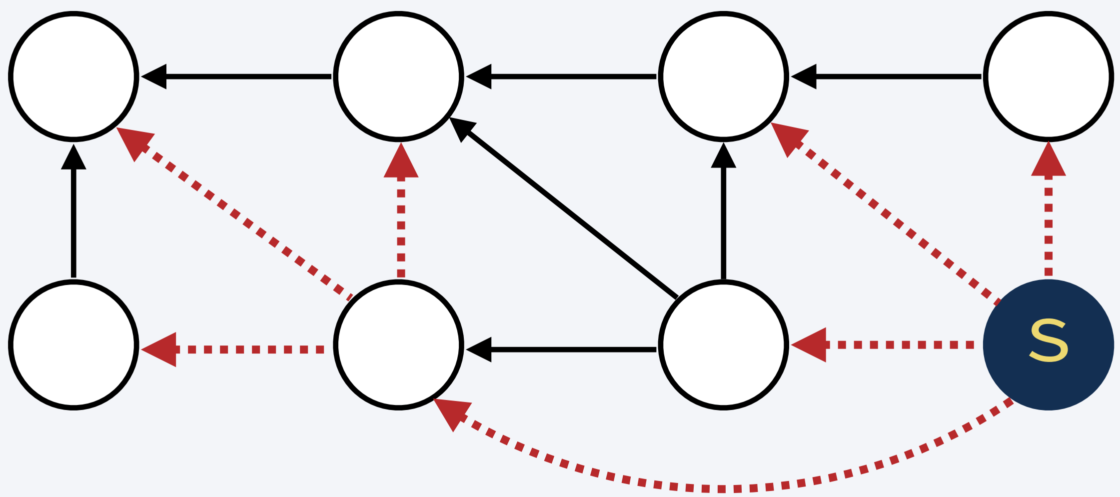


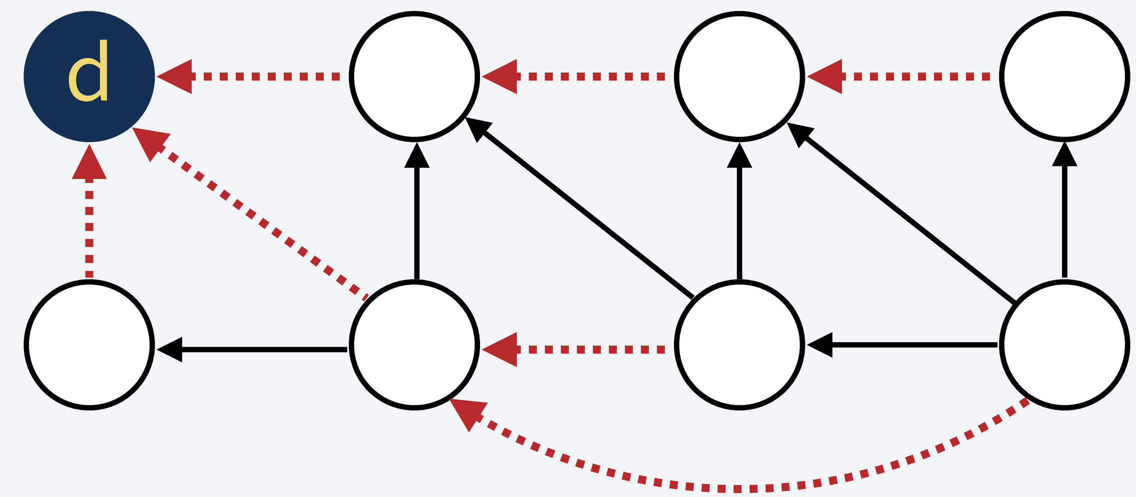


SSSP reduces to SDSP

- Create $G^T$, a transpose of $G$.
- Set $s$ to $d$ and run **SSSP** on $G^T$.
- Transpose the shortest paths tree.

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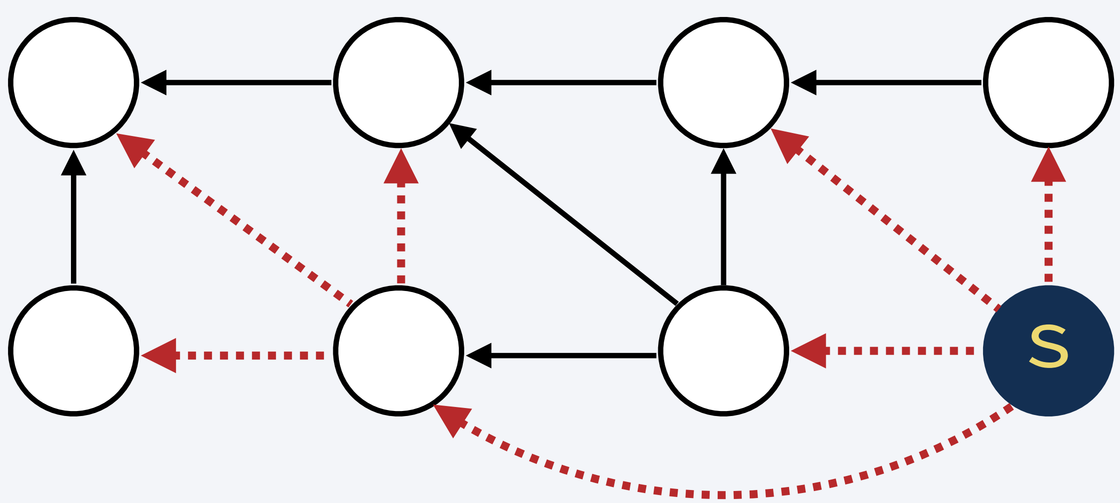


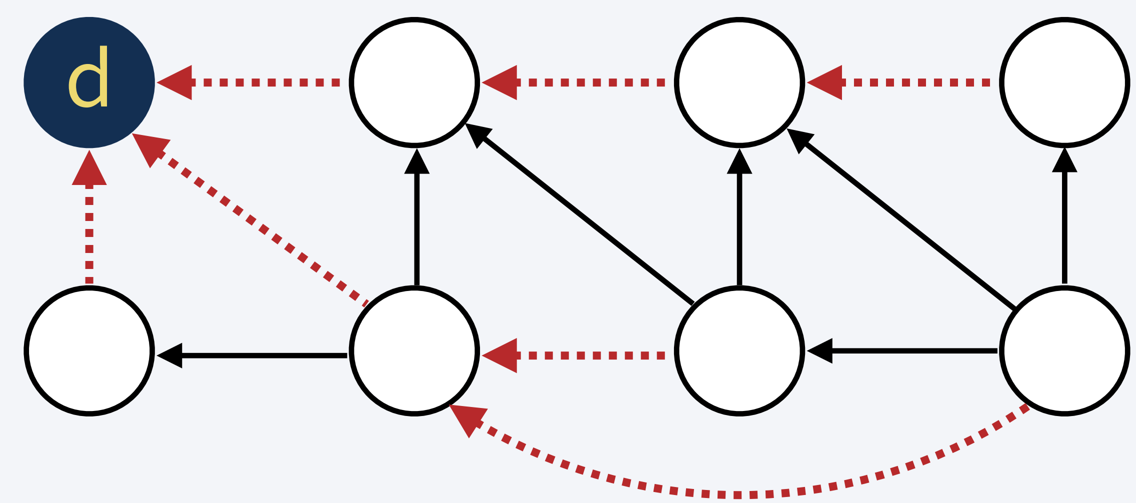


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Running Time. $O(E \log V) + O(E + V)$

reduction (transposing)

SSSP (using Dijkstra's algorithm, assuming the graph is cyclic and has non-negative weights)

# Quiz # 1

Suppose there is a proof that no computer can solve problem $X$.

How can we prove that a problem $Y$ is also impossible to solve?

**A.** Show that $X$ reduces to $Y$.

**B.** Show that $Y$ reduces to $X$.

**C.** Computers can solve any problem. It is only that *we* might not be clever enough to come up with an algorithm!

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- **D.** It depends.

$X$ reduces to $Y$

We can use $Y$ to solve $X$.

If $Y$ is solvable:
$X$ is also solvable (contradiction!)

$Y$ reduces to $X$

We can use $X$ to solve $Y$.

While $X$ is unsolvable, there might be another way for solving $Y$ not using $X$.

# Reductions (Examples)

optional

**TOTALITY**

Does a given program $P$ terminate on all possible inputs?
(never enters an infinite loop!)

**EQUIVALENCE**

Given two programs $P_1$ and $P_2$. Do these two programs produce the same output for every input?
(i.e. are they equivalent?)

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- Use `EQUIVALENCE` to check if $P_1$ and $P_2$ are equivalent.
  If they are equivalent, $P$ terminates on all input. If they are not, the only possibility is that $P$ does not terminate on some input (since the output of $P_1$ and $P_2$ is always the same).

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# Reductions (Examples)

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**PAIR**

Given lists $L_1$ and $L_2$ of size $N$, pair the min in $L_1$ with the min in $L_2$, the next min in $L_1$ with the next min in $L_2$, etc.

**SORT**

Given a list of elements, sort them in non-decreasing order.

Example.

$L_1$ = [13, 7, 3, 1, 11, 2]

$L_2$ = [2, 8, 6, 4, 10, 0]

PAIR = [1-0, 2-2, 3-4, 7-6, 11-8, 13-10]

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### SORT reduces to PAIR

- Let $L_1$ be the list to be sorted.
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- Extract the sorted version of $L_1$ from the result of applying PAIR on $L_1$ and $L_2$.

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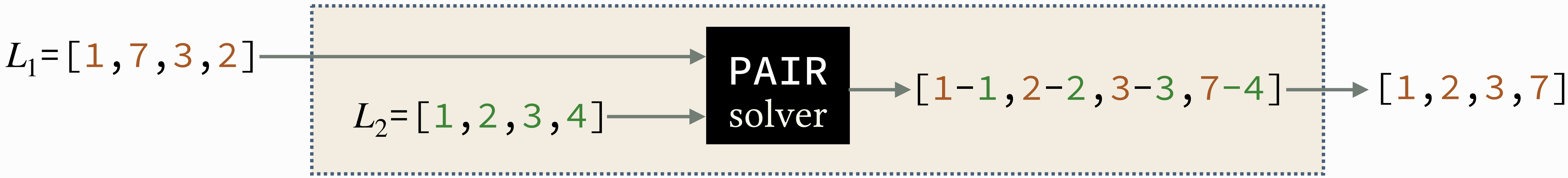


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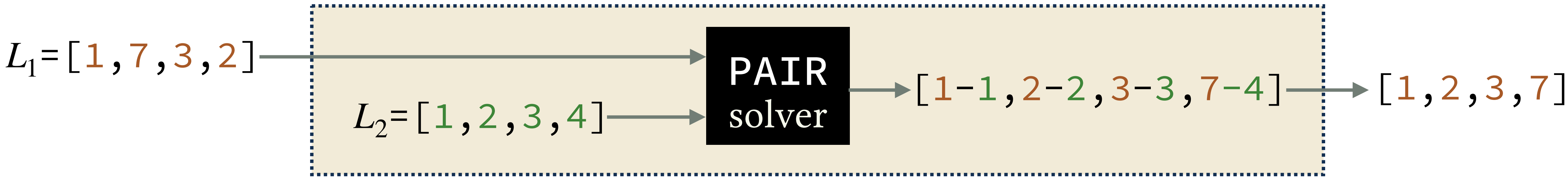


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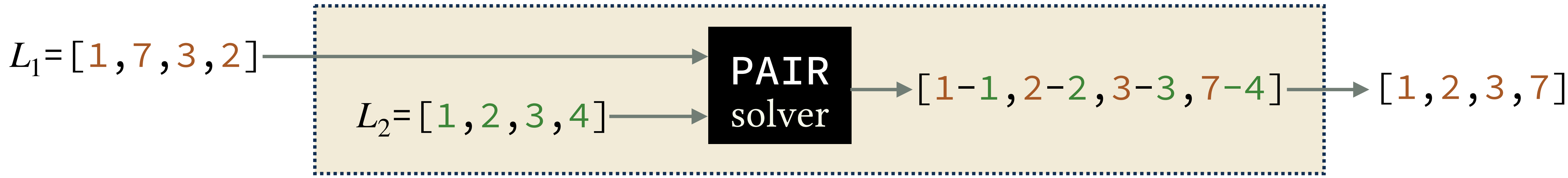


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- The reduction from `SORT` to `PAIR` requires only $\Theta(N)$ amount of work (creating $L_2$ and extracting the result)

### `PAIR` reduces to `SORT`

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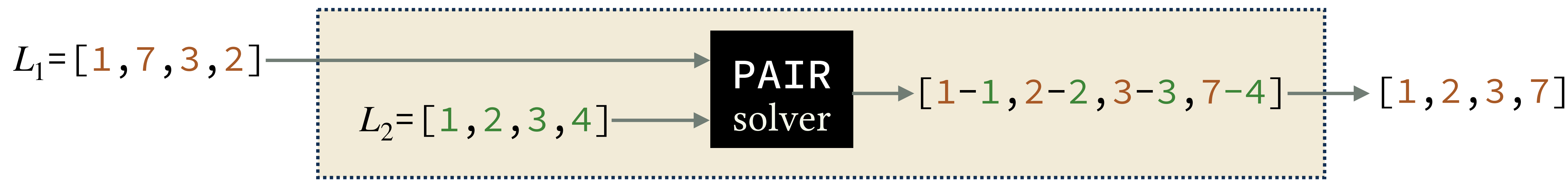


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- The reduction from SORT to PAIR requires only $\Theta(N)$ amount of work (creating $L_2$ and extracting the result)
- PAIR must require $\Omega(N \log N)$ compares in the worst case.
  Otherwise, the $\Omega(N \log N)$ lower bound for SORT is not correct (contradiction!)

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## NEVER FORGET

If A is hard to solve and
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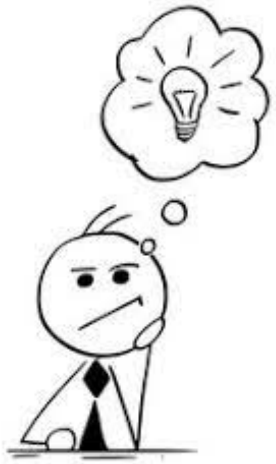

What does it mean for a problem to be hard anyway?

# A fine line Between *Hard* and *Easy* Problems

Shortest Paths on unweighted graphs

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Shortest Paths on unweighted graphs — `O(E+V) using BFS`

# A fine line **Between *Hard* and *Easy* Problems**

Shortest Paths on unweighted graphs ——— `O(E+V) using BFS`

Shortest Paths on weighted DAGs

# A fine line **Between *Hard* and *Easy* Problems**

- 😏 Shortest Paths on unweighted graphs —— `O(E+V) using BFS`
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- 0-1 Knapsack Problem — `NO KNOWN POLYNOMIAL TIME ALGORITHM EXISTS!`

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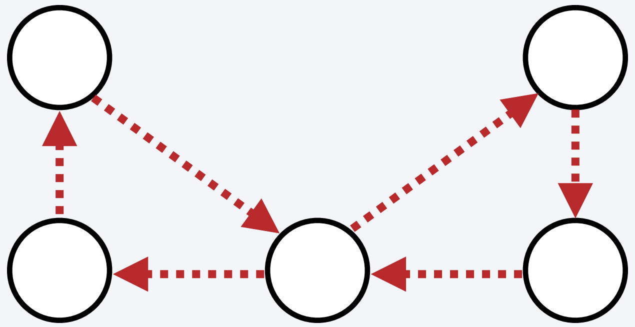
- Change Making for canonical coin systems — has an efficient greedy algorithm
- Change Making for arbitrary coin systems

NO KNOWN POLYNOMIAL TIME ALGORITHM EXISTS!

# A fine line **Between *Hard* and *Easy* Problems**



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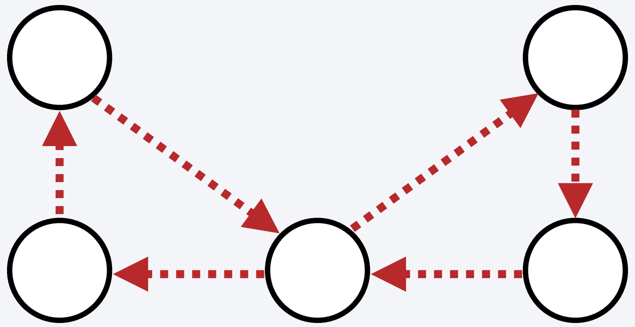

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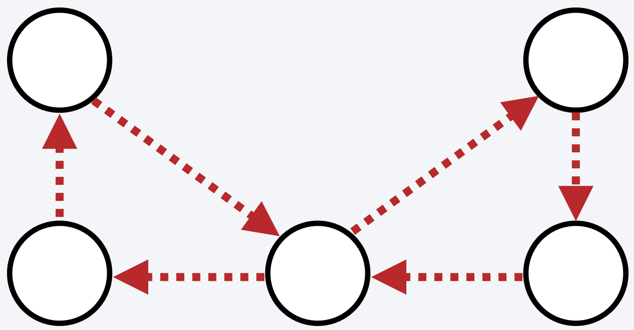

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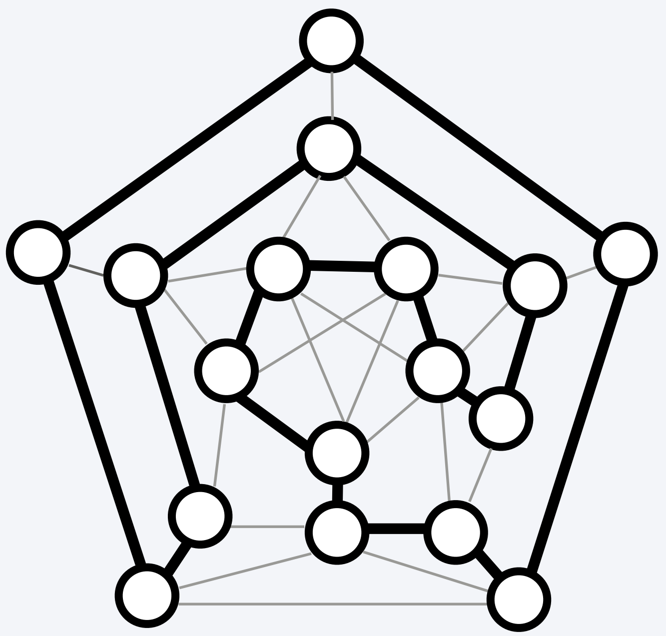

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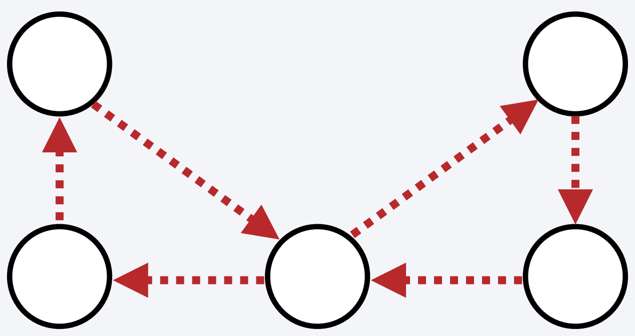

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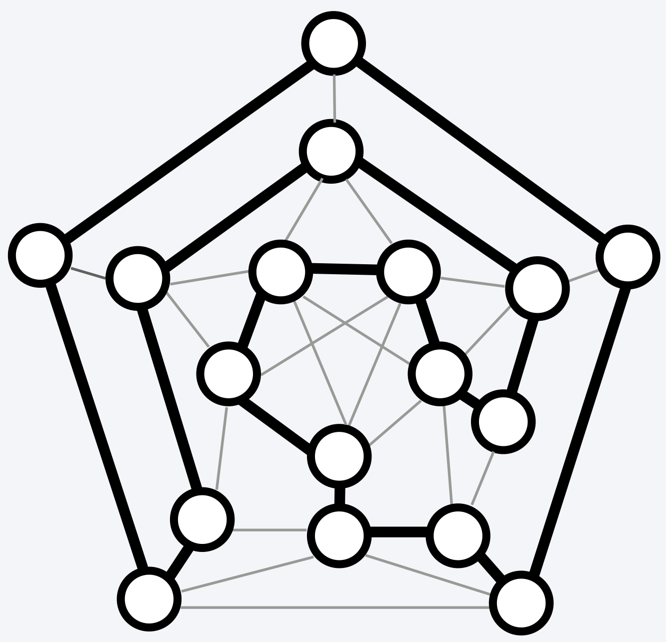

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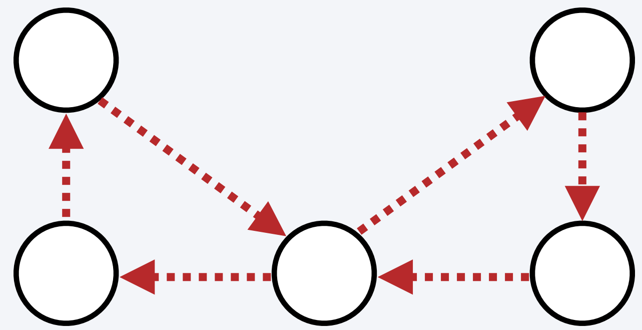

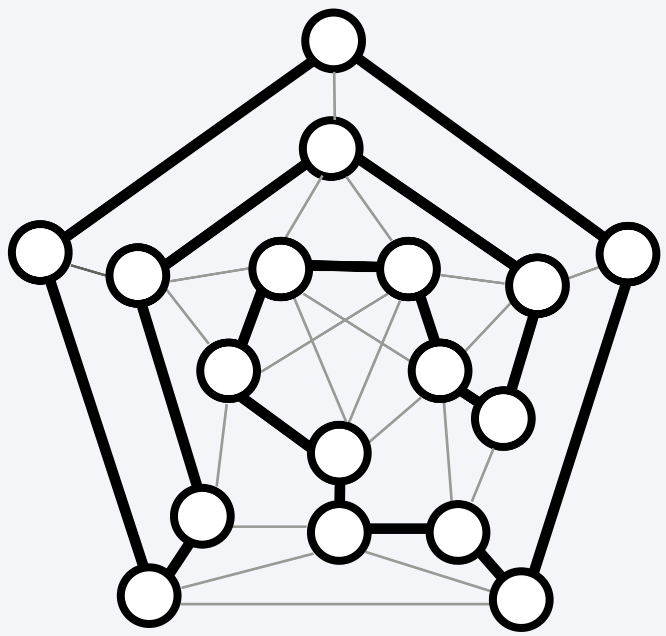

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`NO KNOWN POLYNOMIAL TIME ALGORITHM EXISTS!`

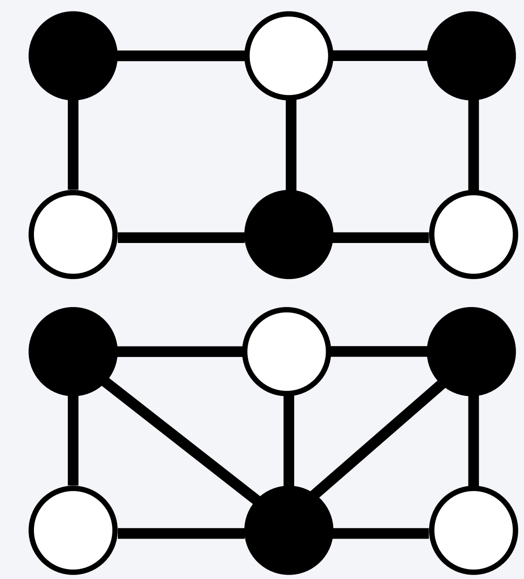
Traveling Salesman Problem (TSP)
Given a complete weighted graph, what is the *shortest Hamiltonian Cycle*?

`NO KNOWN POLYNOMIAL TIME ALGORITHM EXISTS!`

# A fine line **Between *Hard* and *Easy* Problems**

Is a graph 2-Colorable?
(can the vertices be colored using 2 colors, such that no two adjacent vertices have the same color?)

Direct solution: True if there is no cycle of odd length
(can be checked using BFT)



Is a graph *k*-Colorable?
(can the vertices be colored using *k* colors or less, such that no two adjacent vertices have the same color?)

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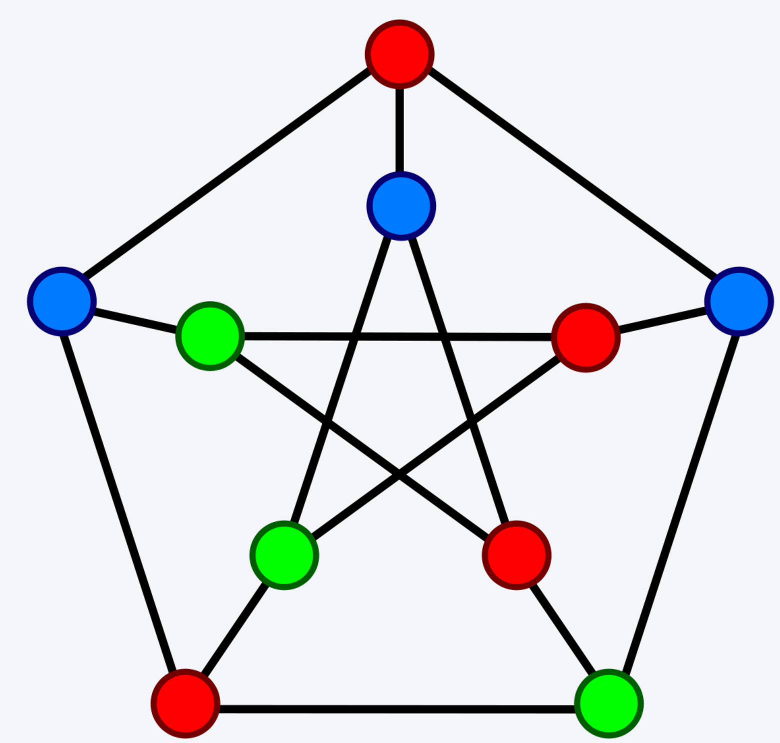

# More Hard Problems

Bin Packing

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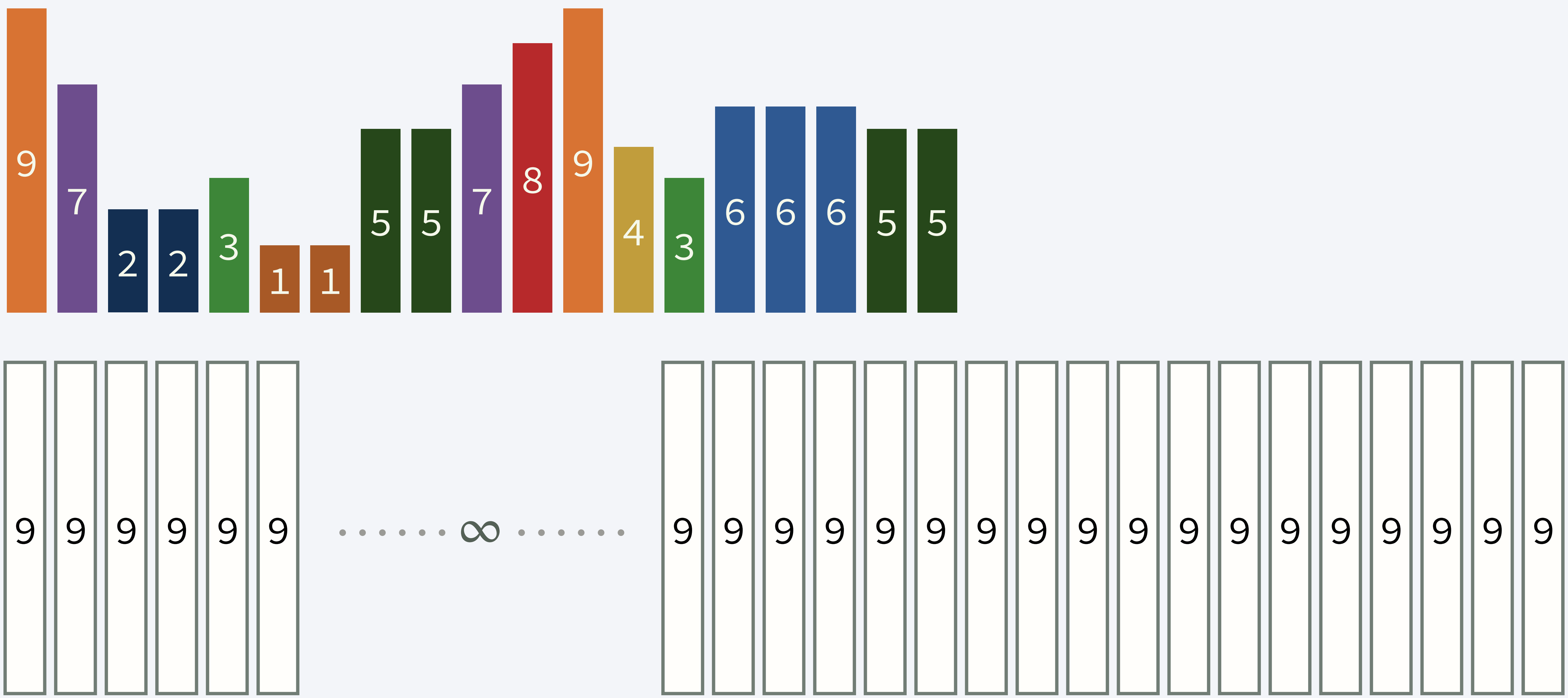

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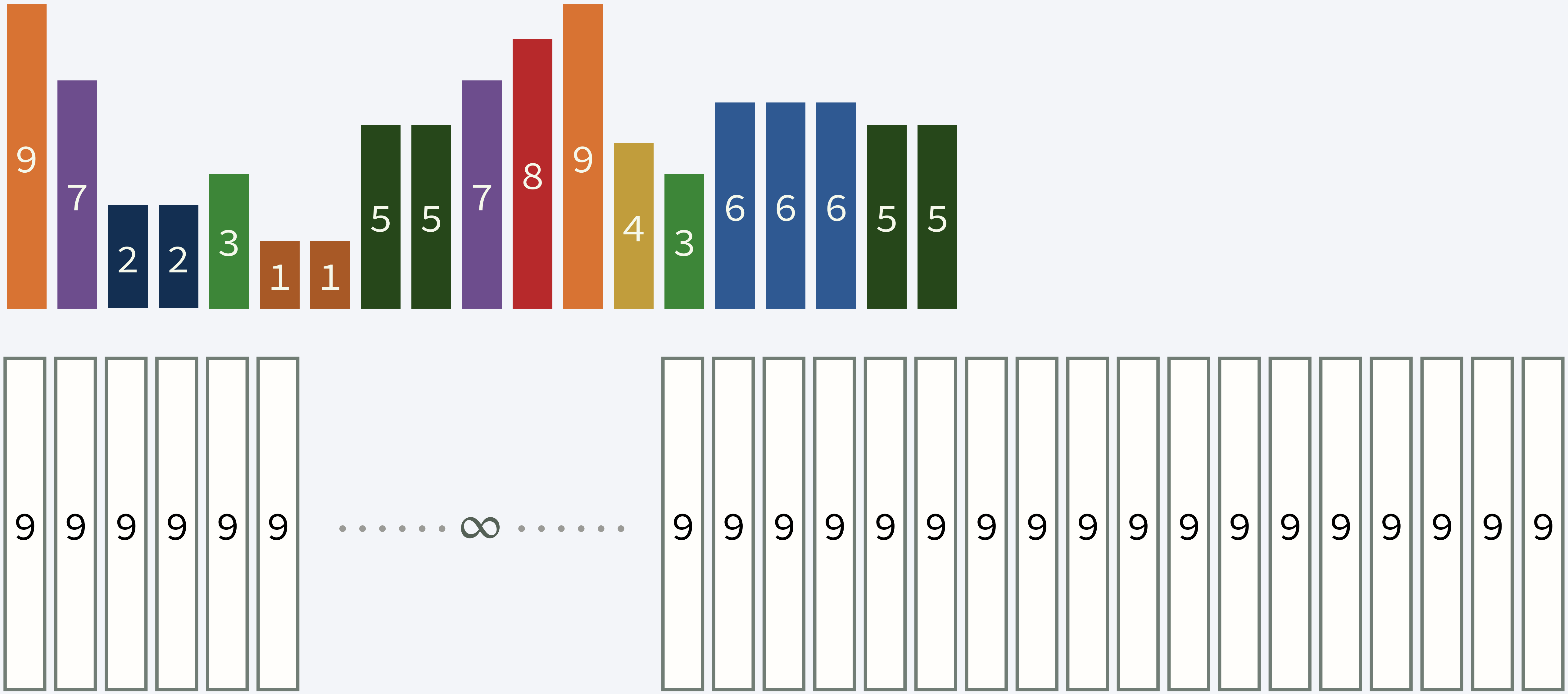

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**Example.** $S$ = {1, 1, 1, 4, 4, 5, 6}, $k$ = 8

Possible Subsets: {1, 1, 1, 5}, {1, 1, 6}, {4, 4} ← min subset

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**YES:** {1, 4} and {2, 3}

$S$ = {1, 2, 3, 4, 5}

**No**

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What is common between finding longest paths in cyclic graphs, 0-1 Knapsack, Subset Sum, Subset Partition, Bin-Packing, TSP and Checking if a Hamiltonian cycle exists?
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I.e. Finding a polynomial time solution to any of them means that all of them have polynomial time solutions!

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Welcome to the

# P *vs* NP

Problem

# Definitions

Optimization problem:

Find the ***best*** solution among a set of feasible solutions.

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Requires a **yes/no** answer.

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**Examples** Traveling Salesman Problem

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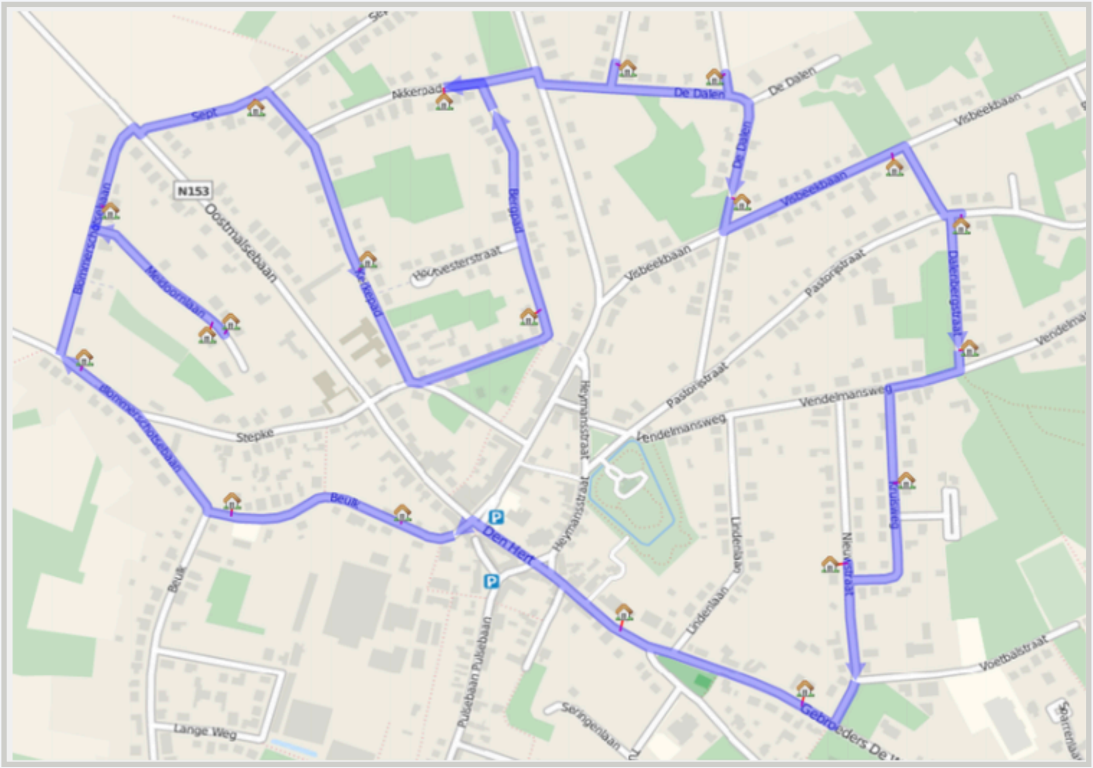
**Decision problem:**
Given a complete weighted graph $G$, does $G$ contain a simple circuit $C$ that visits each node exactly once such that the total cost of the edges in $C$ is *less than or equal to some threshold $T$*?



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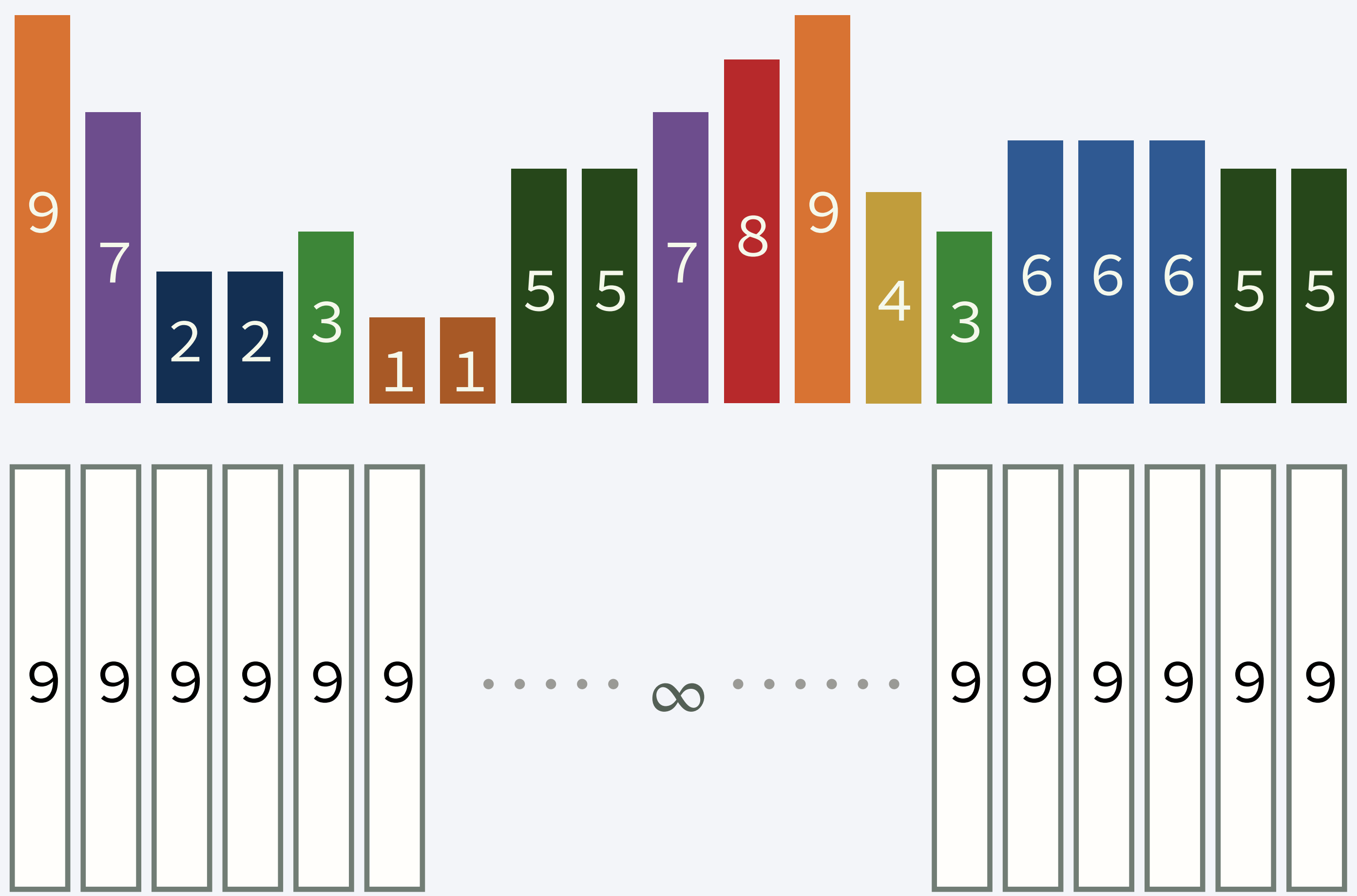

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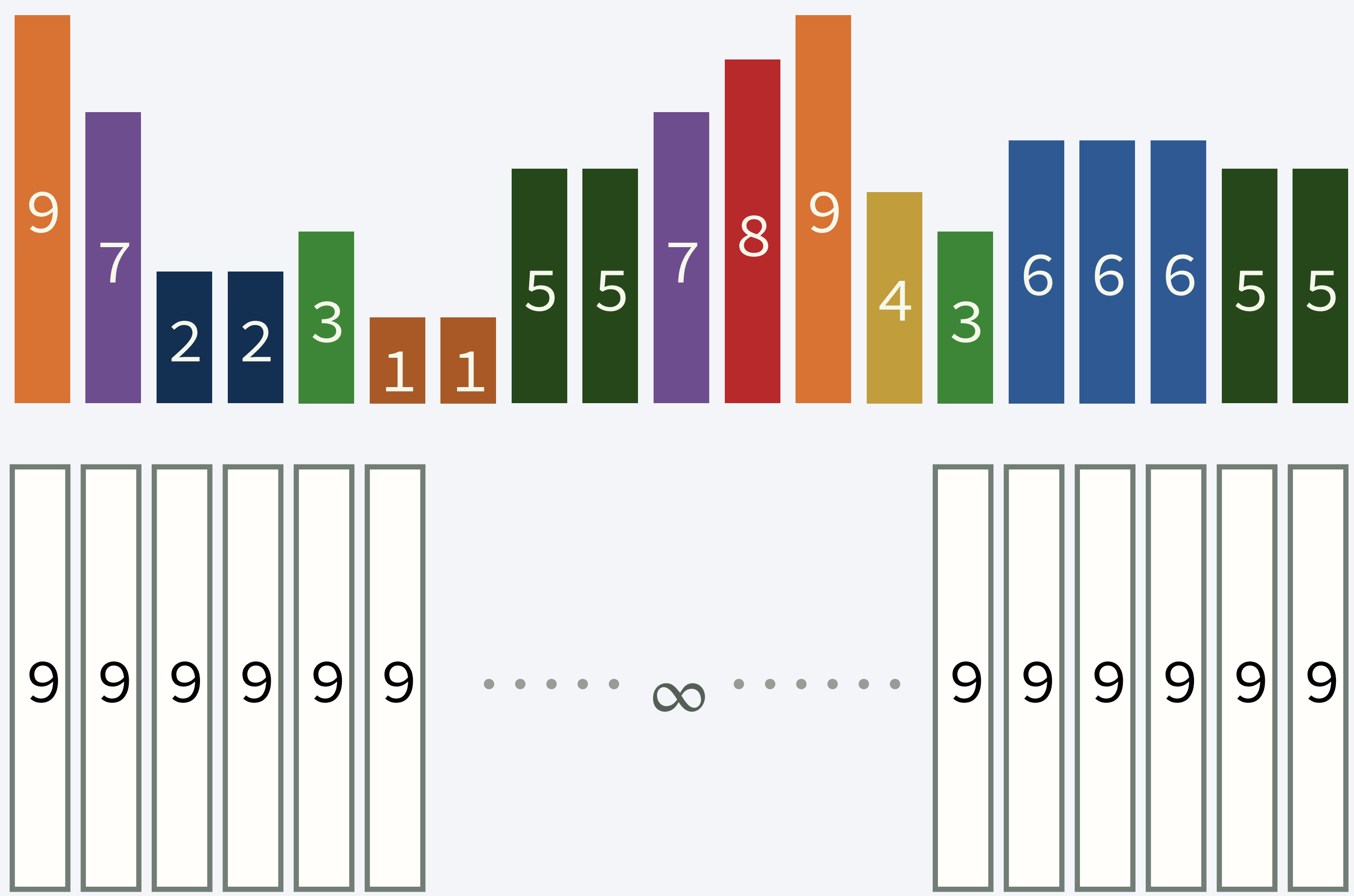
Decision problem:
Can the objects fit in *less than k bins* ?



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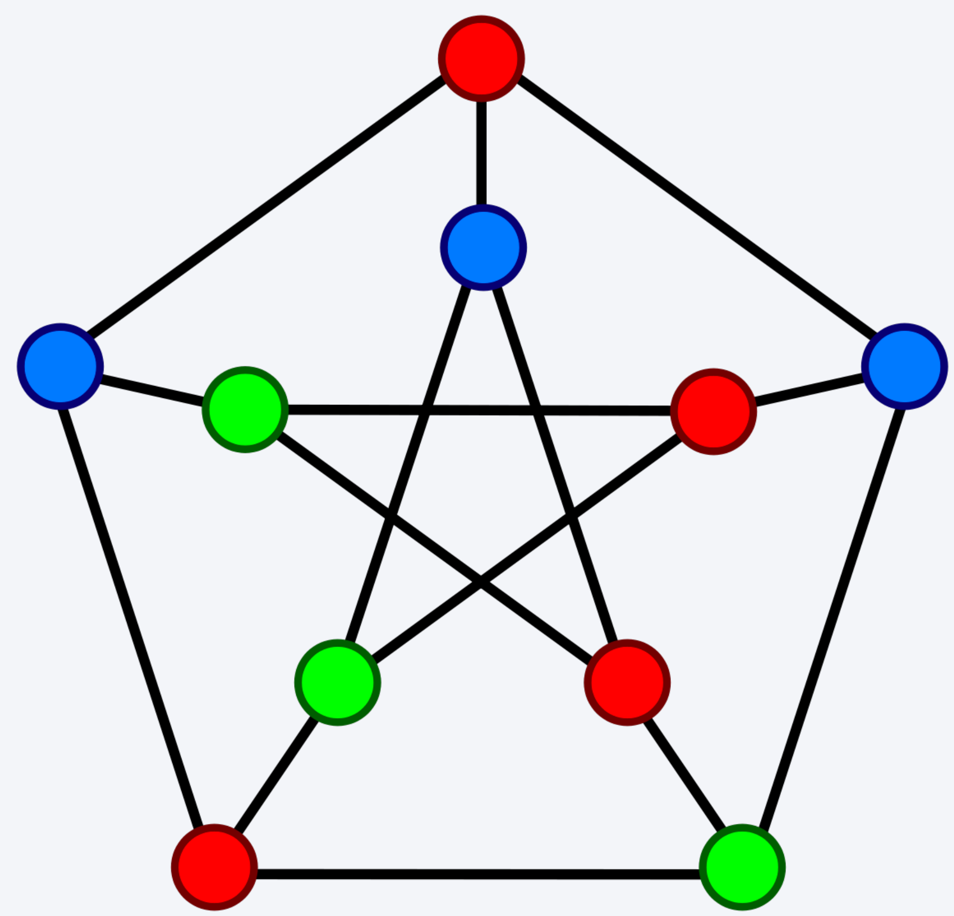

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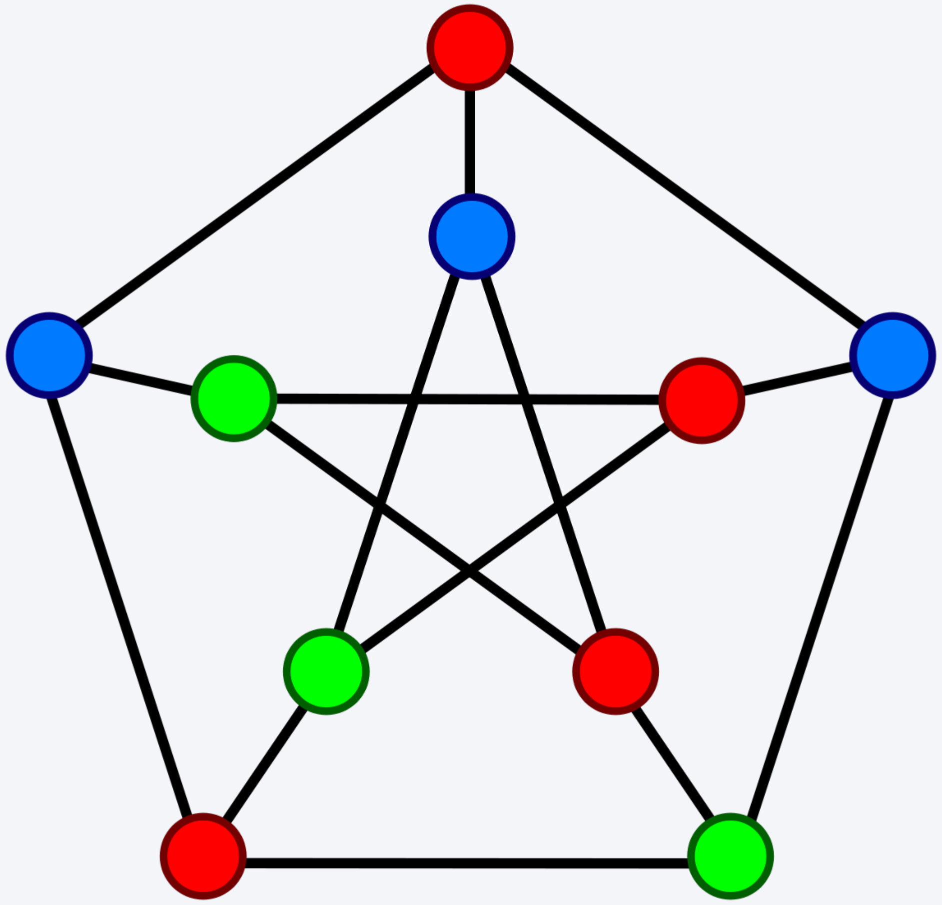
Decision problem:
Can the vertices be properly colored *in K or fewer* colors such that adjacent vertices are not assigned the same color?



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## Examples — Subset Sum

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Given a multi-set $S$ of integers and an integer $k$, find a *minimum* subset of $S$ whose elements sum up to exactly $k$.

**Example.**

$S$ = {1, 1, 1, 4, 4, 5, 6}, $k$ = 8

Possible Subsets: {1, 1, 1, 5}
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{4, 4} ⟵ *minimum*

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Decision problem:
*Does S contain a subset* whose elements sum up to exactly $k$?

**Example.**

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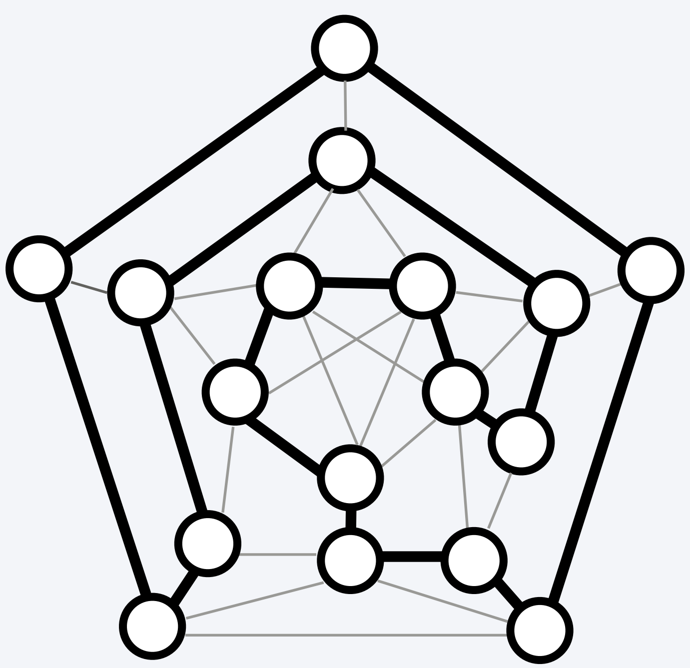
Decision problem:
Is there a cycle that visits each vertex in the graph once?



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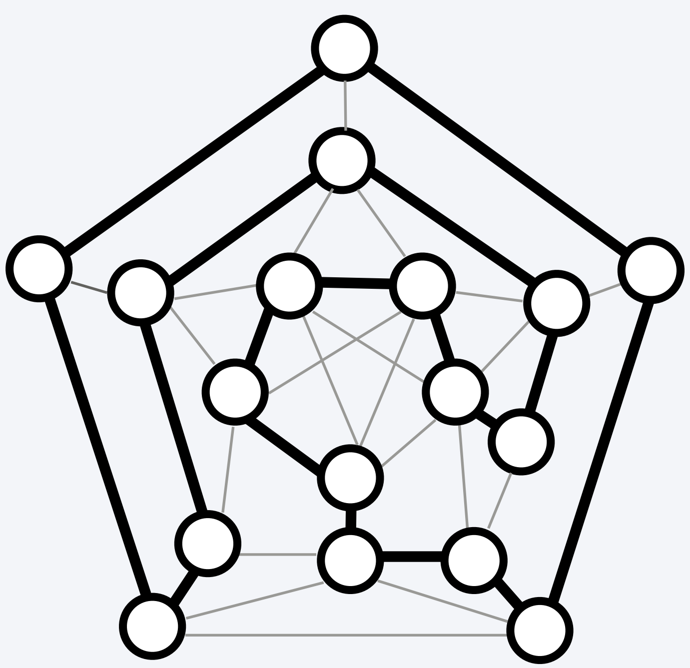

Examples Subset Partition

Decision problem:
Given a set $S$ of integers, Can we partition $S$ into two subsets of exactly the same size?

**Example.** $S$ = `{1, 2, 3, 4}`
**`YES:`** `{1, 4} and {2, 3}`

$S$ = `{1, 2, 3, 4, 5}`
**`No`**

# Quiz # 2

Given a solver for the optimization version of TSP, how can we solve the decision version?

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**Answer**. If we know the length of the shortest tour $L$, we can very easily answer the question *Is there a tour of length less than* $T$ as follows:

If $L \geq T$ : There is no tour of length less than $T$.
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Given a solver for the decision version of TSP, how can we solve the optimization version?

**Answer**.

- Compute a bound $B$ for the length of the shortest tour (e.g. the sum of the edge weights int he graph, or $V \times$ the largest weight)
- Use binary search to find the length of the shortest tour:

  Use the solver of the decision problem to answer the question:
  *Is there a tour of length less than $B/2$* ?

  Eliminate the left or right half based on the answer and repeat.

# Quiz # 3

If the decision version of a problem is hard, does this imply that the optimization version is also hard?

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If the decision version of a problem is hard, does this imply that the optimization version is also hard?

**Answer**. Yes.

The decision version is no harder (as hard or easier) than the optimization version.

To discuss and prove hardness,
we will consider only *decision problems*!

# Definitions (Complexity Classes)

**Class P.**

A decision problem is in P if it is solvable in polynomial time
(i.e. in $O(n^c)$, where $n$ is the input size and $c$ is a constant)

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## Examples

- Given a list of integers $L$ and an integer $K$:
  - is $K$ in $L$?
  - Is there an integer in $L$ that is greater than $K$?
  - Do any two numbers in $L$ sum to $K$?
- Given a permutation of elements $P$:
  - is $P$ sorted in ascending order?
  - is $P$ a palindrome?
- Given a graph $G$:
  - Is there a spanning tree whose sum of edge weights is less than $T$?
  - Is there a path between $v$ and $w$ in a graph $G$ less than $T$?
  - Is there a cycle in the graph?
  - Is the graph connected?
- Given a set of activities, can we schedule $X$ activities without overlap?

etc.

# Quiz # 4

Which of the following problems are *not* in **P** ?

**A.** Traveling Salesman Problem.

**B.** 0-1 Knapsack.

**C.** Bin-Packing.

**D.** All of the above.

I don't know.

# Quiz # 4

Which of the following problems are *not* in **P** ?

**A.** Traveling Salesman Problem.

**B.** 0-1 Knapsack.

**C.** Bin-Packing.

**D.** All of the above.

We don't know.

A problem is in **P** if it has a polynomial time solution.

A problem is *not* in **P** if there is a proof that it does not have a polynomial time solution.

No one proved that these problems do not have polynomial time solutions!

# Definitions (Complexity Classes)

## Class **P**.

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## Class **NP**.

A decision problem is in NP if it is verifiable in polynomial time.
(Given an instance $I$ or a problem $P$ and a witness $W$ for the solution, can we verify in polynomial time if $W$ proves that the answer for $I$ is yes?)

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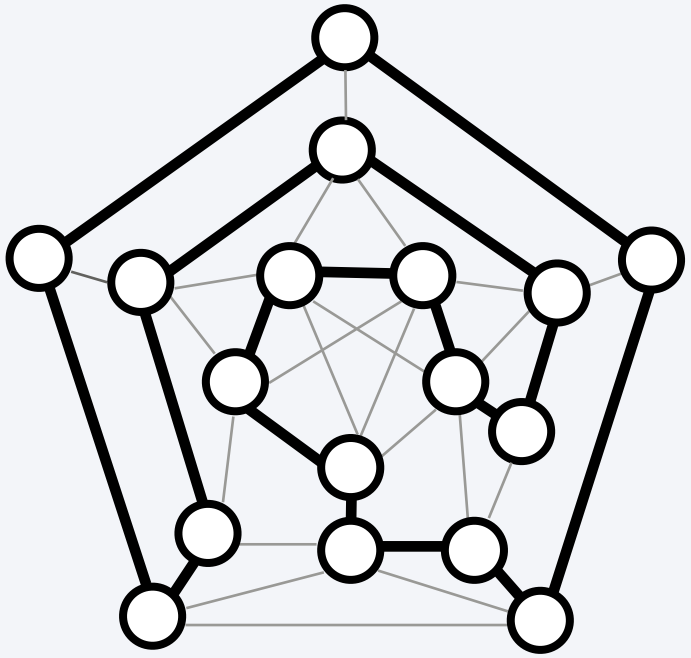
**Example** Is there A `HAMILTONIAN` Cycle?

Given a graph $G$, and a path $C$ (a witness), can we verify in polynomial time if $C$ is a hamiltonian cycle?



Yes!

1. Check that the first and last vertices are the same.
2. Check that no vertex repeats.
3. Check that the path has exactly $V$ edges.

# Definitions (Complexity Classes)

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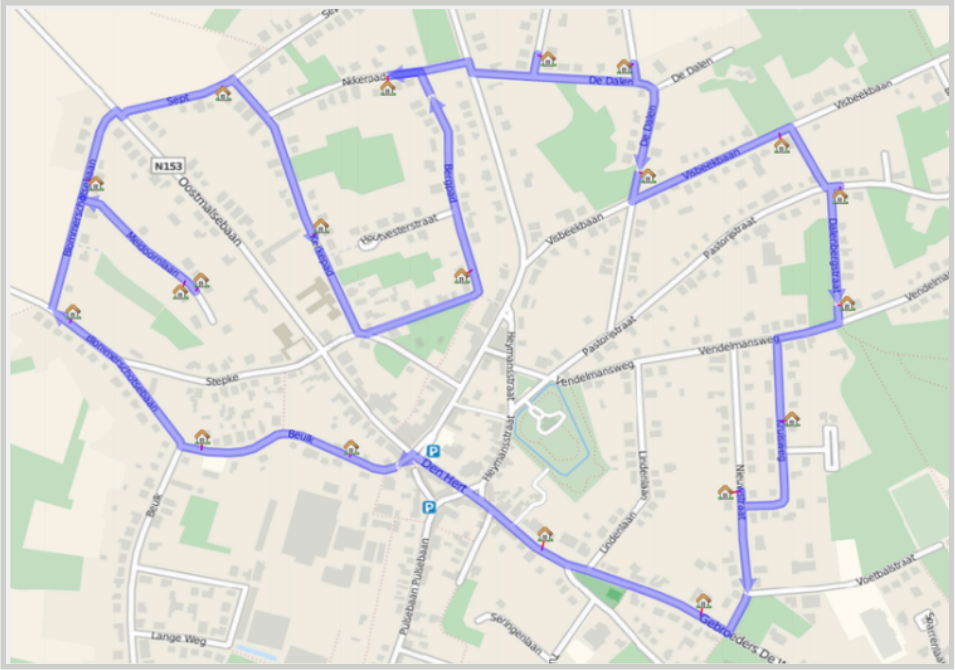
**Example** TSP is in NP

Given a graph $G$, a length $L$, and a path $C$ (a witness), can we verify in polynomial time if $C$ is a hamiltonian cycle of length less than $L$?



Yes!

1. Check that $C$ is a Hamiltonian cycle.
2. Check that the sum of the edge weights is less than $L$.

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**Example** SUBSET-SUM is in NP

Given a multi-set $S$, two integers $K$ and $L$, and a subset $H$ of $S$ (a witness), can we verify in polynomial time if $|H| \leq K$ and that its elements sum to $L$?

Yes!

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Yes!

**Example** `SUBSET-PARTITION`

Given a multi-set $S$, two subsets $H_1$ *and* $H_2$ of $S$ (a witness), can we verify in polynomial time if $|H_1| + |H_2| = |S|$ and that the sum of the elements in $H_1$ = the sum of the elements in $H_2$?

Yes!

# Quiz # 5

Every problem that is in **P** is also in **NP**.

- **A.** True.
- **B.** False.
- 

We don't know.

# Quiz # 5

Every problem that is in **P** is also in **NP**.

- **A.** True.
- **B.** False.
- 

We don't know.

If a problem is solvable in polynomial time, it is also verifiable in polynomial time.

We can always solve the problem to verify a given witness!

# Quiz # 6

Every problem that is in **NP** is also in **P**.

**A.** True.

**B.** False.

We don't know.

# Quiz # 6

Every problem that is in **NP** is also in **P**.

**A.** True.

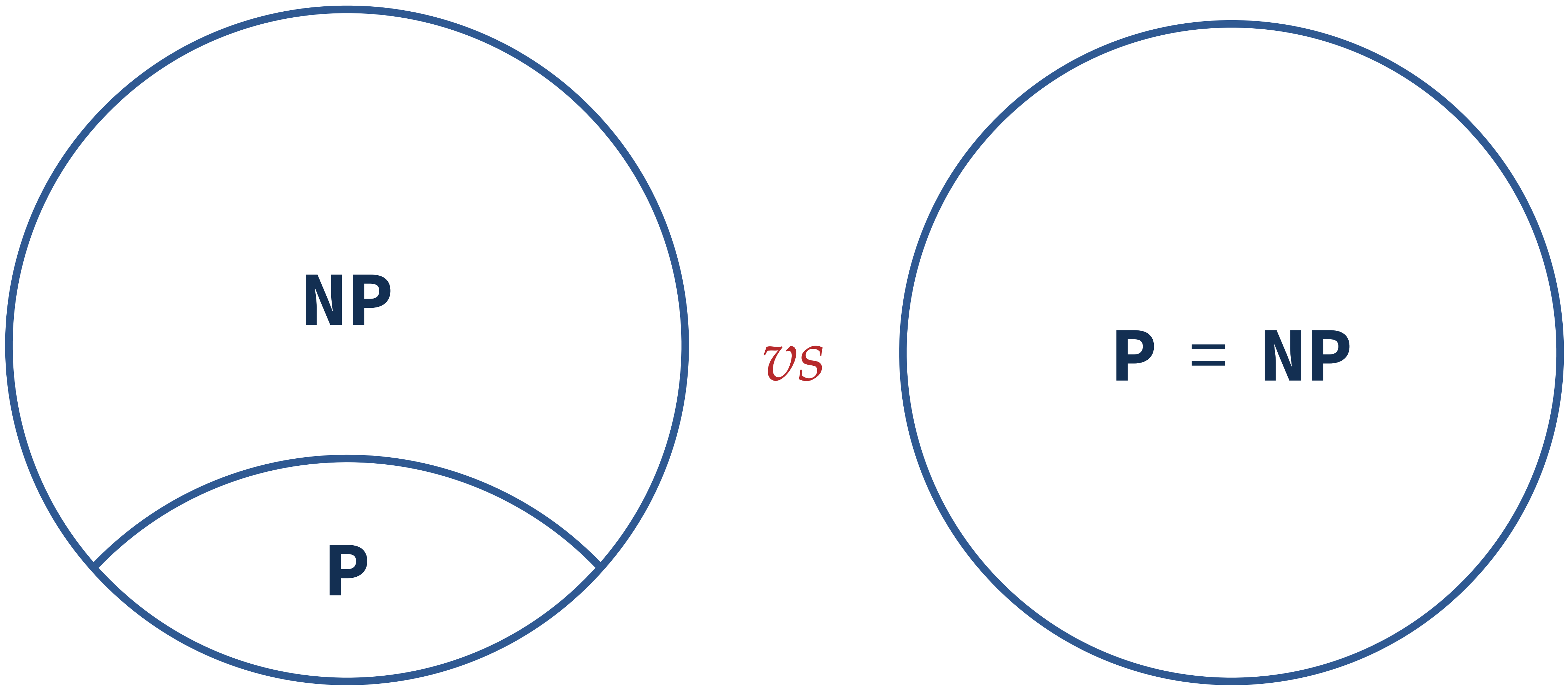
**B.** False.

We don't know.

Does easy verification imply that finding a solution is also easy?

- No one knows!
- No one yet found a problem that is in NP but is not in P !
- This is a $1,000,000 question!

# Two Possible World Views




No one knows which is true!

# Quiz # 7

What are examples of problems that are *not* in **NP**?

# Quiz # 7

What are examples of problems that are *not* in **NP**?

**Example 1.** Given a program $P$ is there an input $I$ that makes $P$ terminate in less than $s$ steps?

**Example 2.** Given a chessboard, is there a move that guarantees black to win?

# What is in a name?

What does **NP** stand for?

- **A.** **N**ot **P**olynomial.
- **B.** **N**o **P**akeup Exam.
- **C.** **N**o **P**roblem.
- **D.** **N**one of the a**P**ove.

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**D.** **N**one of the a**P**ove.

**NP** stands for: **N**on-deterministically **P**olynomial.
I.e. Can be solved using a non-deterministic machine in polynomial time.

Assume that TM is a machine that can guess and verify an infinite number of solutions all at the same time (call TM a *non-deterministic* machine).

If a problem is verifiable in polynomial time, TM can solve the problem by guessing all the possible solutions and verifying them at once (in polynomial time!)

# Definitions (Complexity Classes)

## Class **P**.

A decision problem is in P if it is solvable in polynomial time
(i.e. in $O(n^c)$, where $n$ is the input size and $c$ is a constant)

## Class **NP**.

A decision problem is in NP if it is verifiable in polynomial time.
(Given an instance $I$ or a problem $P$ and a witness $W$ for the solution, can we verify in polynomial time if $W$ proves that the answer for $I$ is yes?)

## Class **NP−Complete**.

A decision problem is NP−Complete if:

- It is in NP.
- All problems in NP reduce to it in polynomial time.

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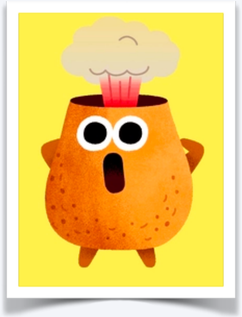
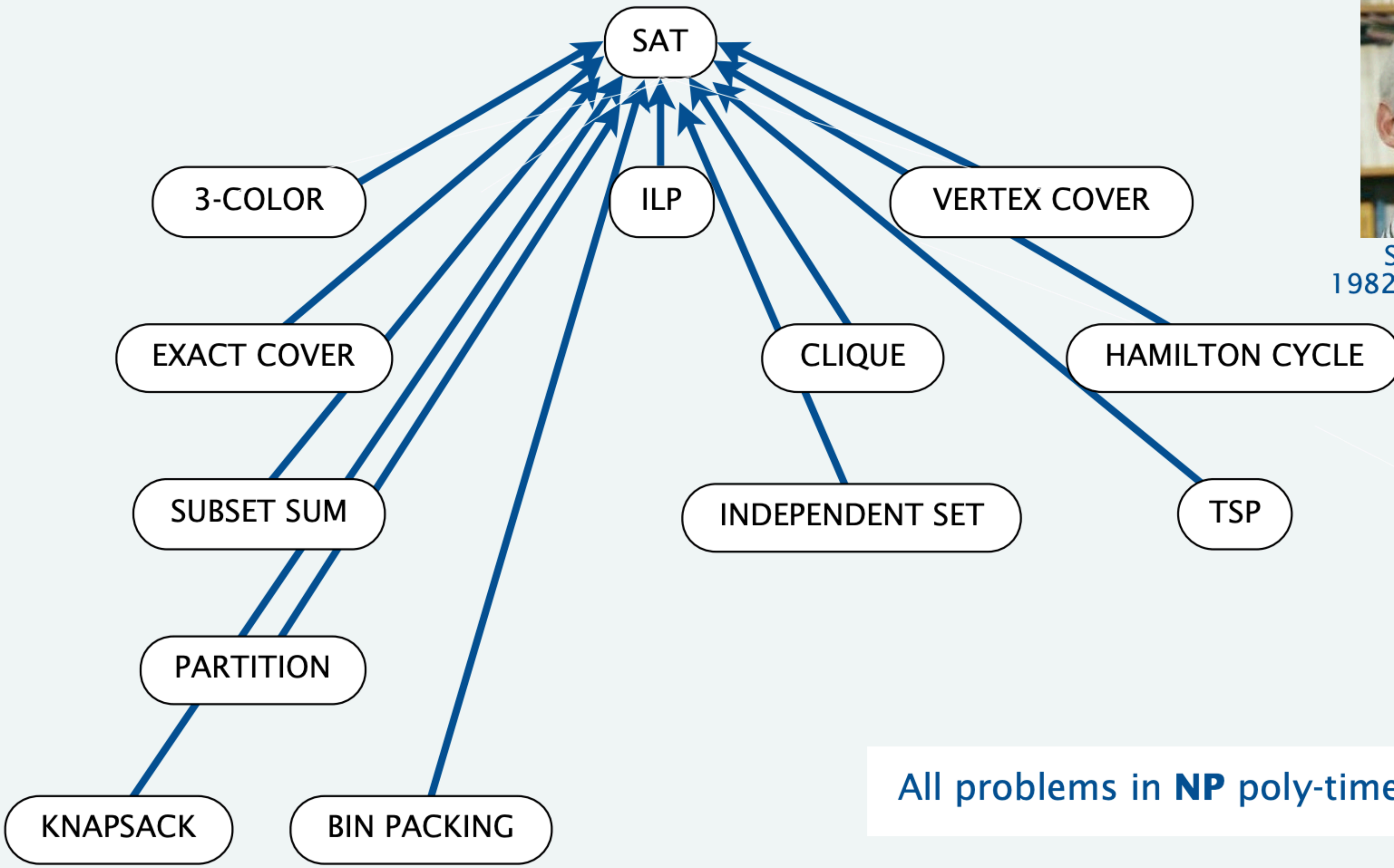
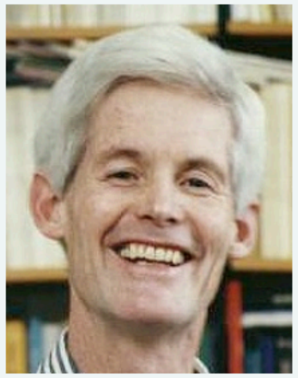


How do we show that *all* problems in **NP** reduce to a certain problem???

# Cook-Levin Theorem (1971)



Steve Cook
1982 Turing Award

Leonid Levin

All problems in **NP** poly-time reduce to SAT.

What is **SAT**?

# Boolean Satisfiability (SAT)

Literal. A Boolean variable or its negation. $x_i$ or $\overline{x_i}$

Clause. A disjunction of literals. $C_j = x_1 \vee \overline{x_2} \vee x_3$

Conjunctive normal form (CNF). A propositional formula $\Phi$ that is a conjunction of clauses. $\Phi = C_1 \wedge C_2 \wedge C_3 \wedge C_4$

SAT. Given a CNF formula $\Phi$, does it have a satisfying truth assignment?

3-SAT. SAT where each clause contains exactly 3 literals
(and each literal corresponds to a different variable).

*adapted from a slide by Kevin Wayne*

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**Example** What values for $x_1$, $x_2$, $x_3$ and $x_4$ satisfy the following formula?

$$\Phi = (\overline{x_1} \vee x_2 \vee x_3) \wedge (x_1 \vee \overline{x_2} \vee x_3) \wedge (\overline{x_1} \vee x_2 \vee x_4)$$

*adapted from a slide by Kevin Wayne*

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**Answer.** $x_1$ = TRUE, $x_2$ = TRUE, $x_3$ = FALSE, $x_4$ = FALSE

*adapted from a slide by Kevin Wayne*

# Boolean Satisfiability (SAT)

Key Facts.

- SAT is in NP.

*adapted from a slide by Kevin Wayne*

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  Given a formula and boolean values for the variables, it is easy to verify if these values satisfy the formula!

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- It is not clear if SAT is also in P.
  - We can try all possible $2^N$ boolean assignments.
  - We don't know if a polynomial time solution exists.

- All problems in in NP reduce to SAT in polynomial time.
  - This is the Cook-Levin Theorem.
  - The details of the proof are beyond the scope of this course.
  - In a nutshell, Cook and Levin showed how any decision problem that is in NP can be converted (in polynomial time) to the problem of satisfying a boolean formula.
    (i.e. a digital circuit can be designed for it that has a polynomial number of gates)

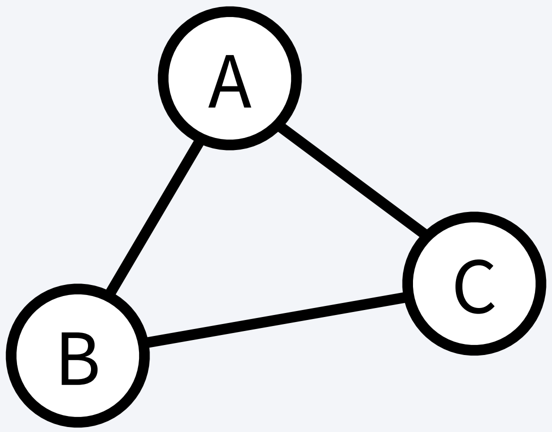
*adapted from a slide by Kevin Wayne*

# Reduction Example

optional

Graph Coloring reduces to SAT in polynomial time.

Assume that the problem is to check if the graph is 2-colorable.



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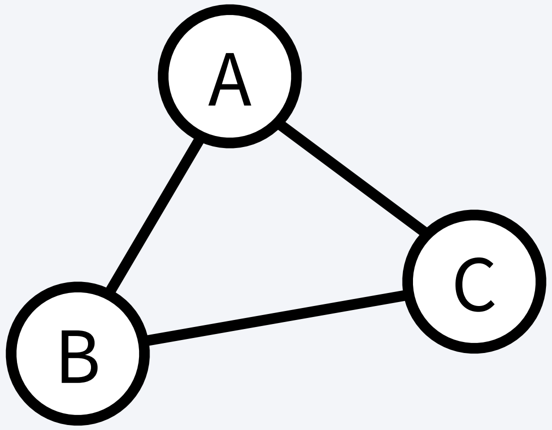


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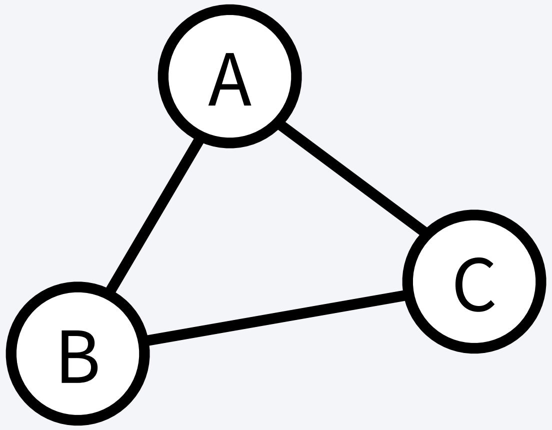


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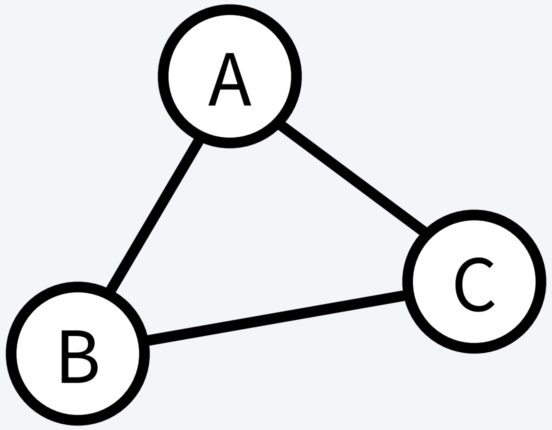


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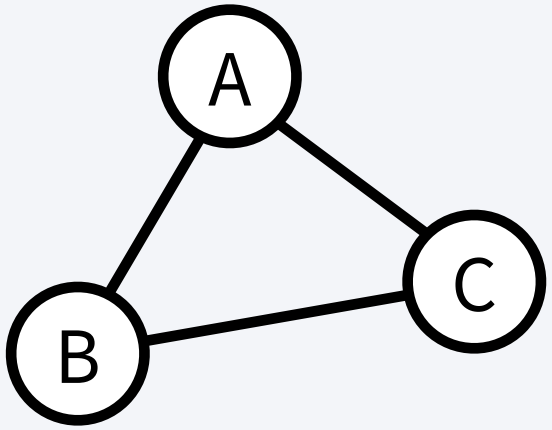


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The graph is 2-colorable if the above boolean expressions are satisfiable!

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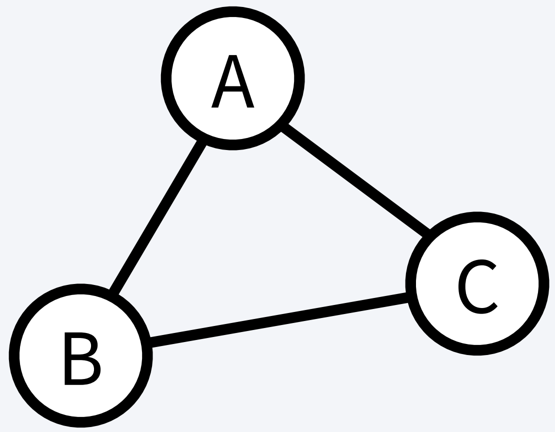


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Can be easily converted to CNF.

The graph is 2-colorable if the above boolean expressions are satisfiable!

# Quiz # 9

How do we show that a problem other than SAT is NP-Complete?

**A.** Be as clever as Cook and Levin and show how all problems in NP reduce to this new problem.

**B.** No need! SAT is the only NP-Complete Problem!

**C.** None of the above.

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**A.** Be as clever as Cook and Levin and show how all problems in NP reduce to this new problem.

**B.** No need! SAT is the only NP-Complete Problem!

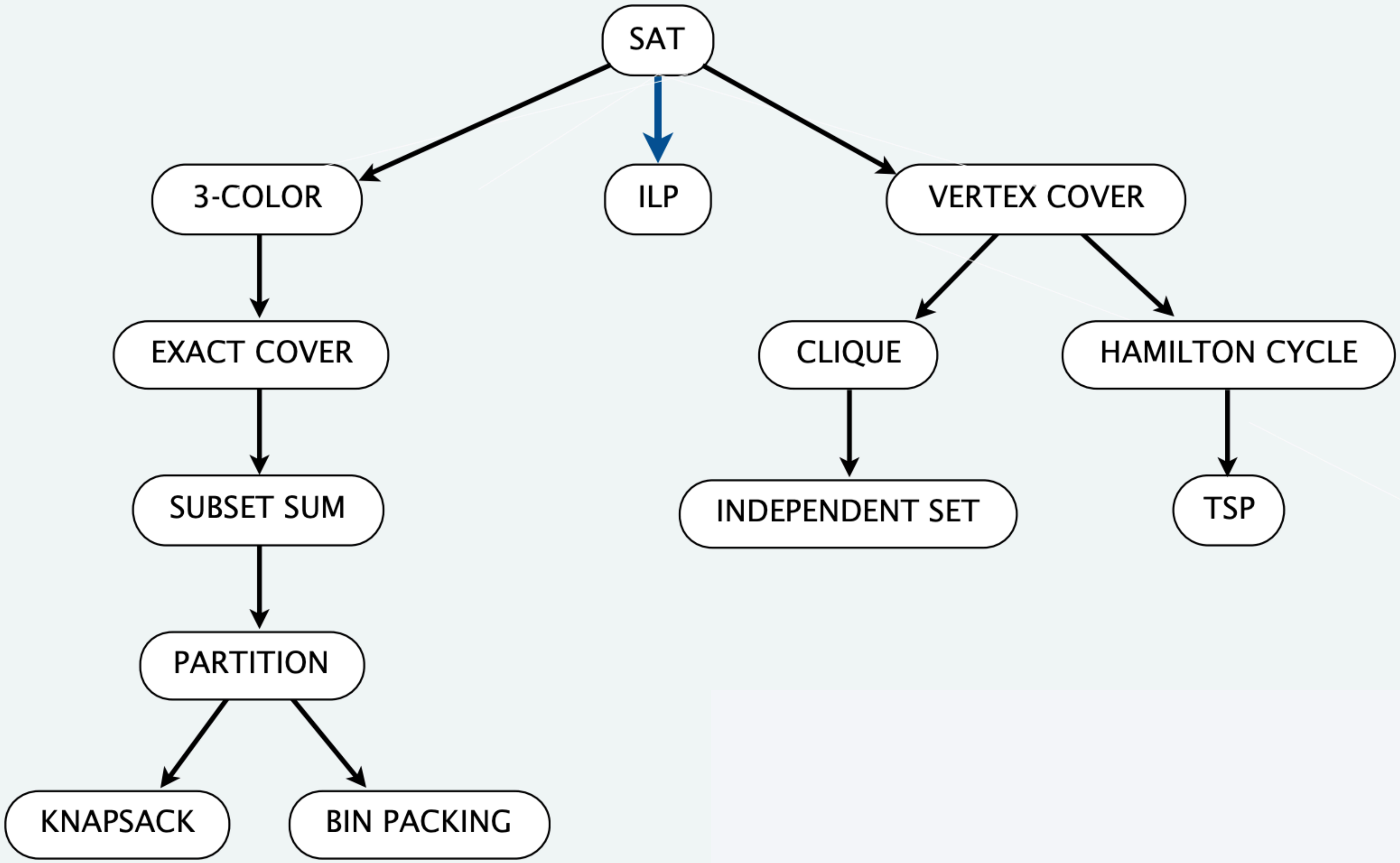
**C.** None of the above.

To show that a problem is NP-Complete:

1. Show that it is in NP.
2. Show that an NP-Complete problem reduces to it in polynomial time!

If all problems in NP poly-time reduce to $A$ and $A$ poly-time reduces to $B$, then all problems in NP poly-time reduce to $B$!

# SAT is not The Only NP-Complete Problem!





Dick Karp
1985 Turing Award

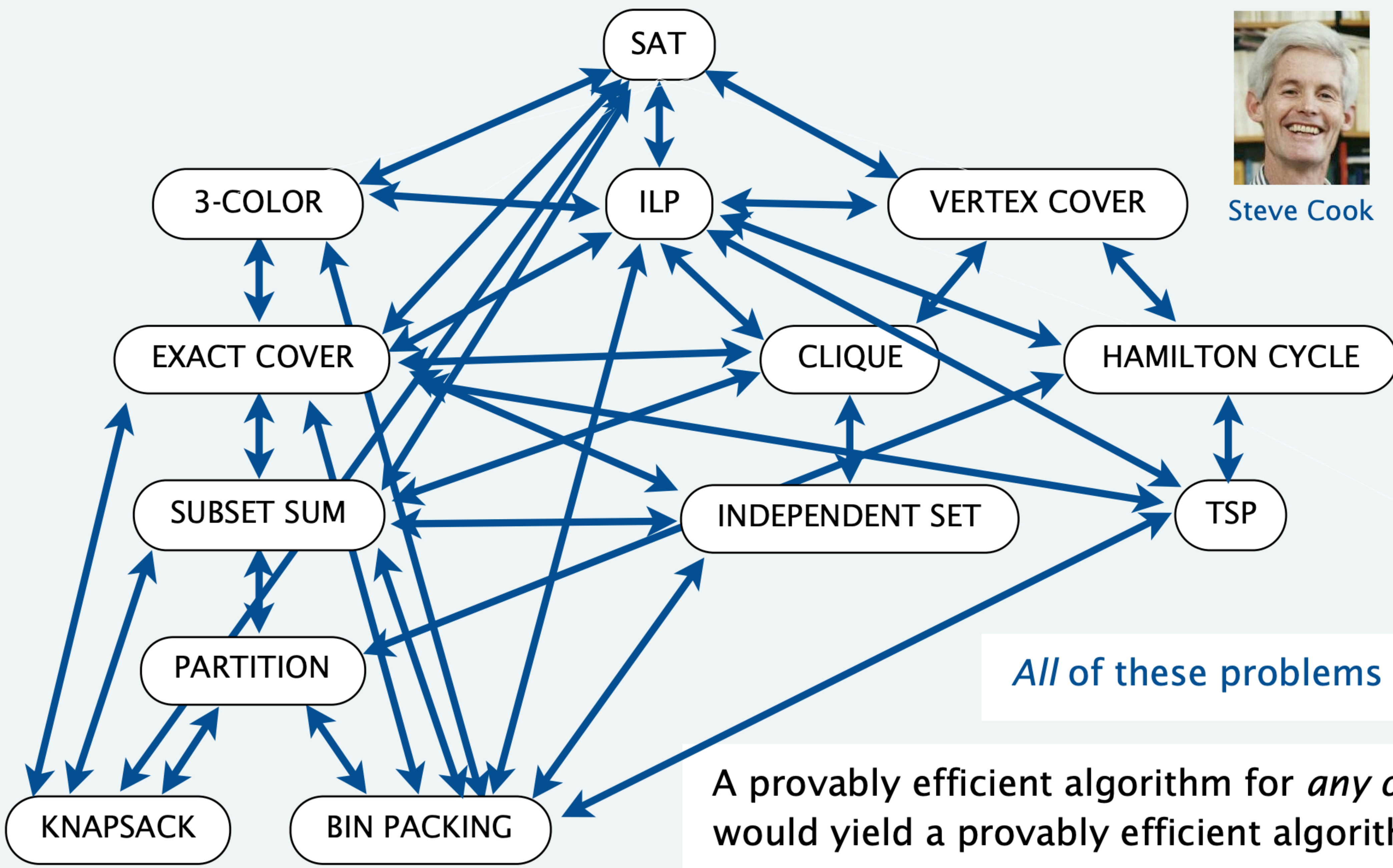
Key Finding. SAT poly-time reduces to many problems!

Implication. All of these problems are NP-Complete!

*slide by Kevin Wayne*

# SAT is not The Only NP-Complete Problem!

*adapted from a slide by Kevin Wayne*

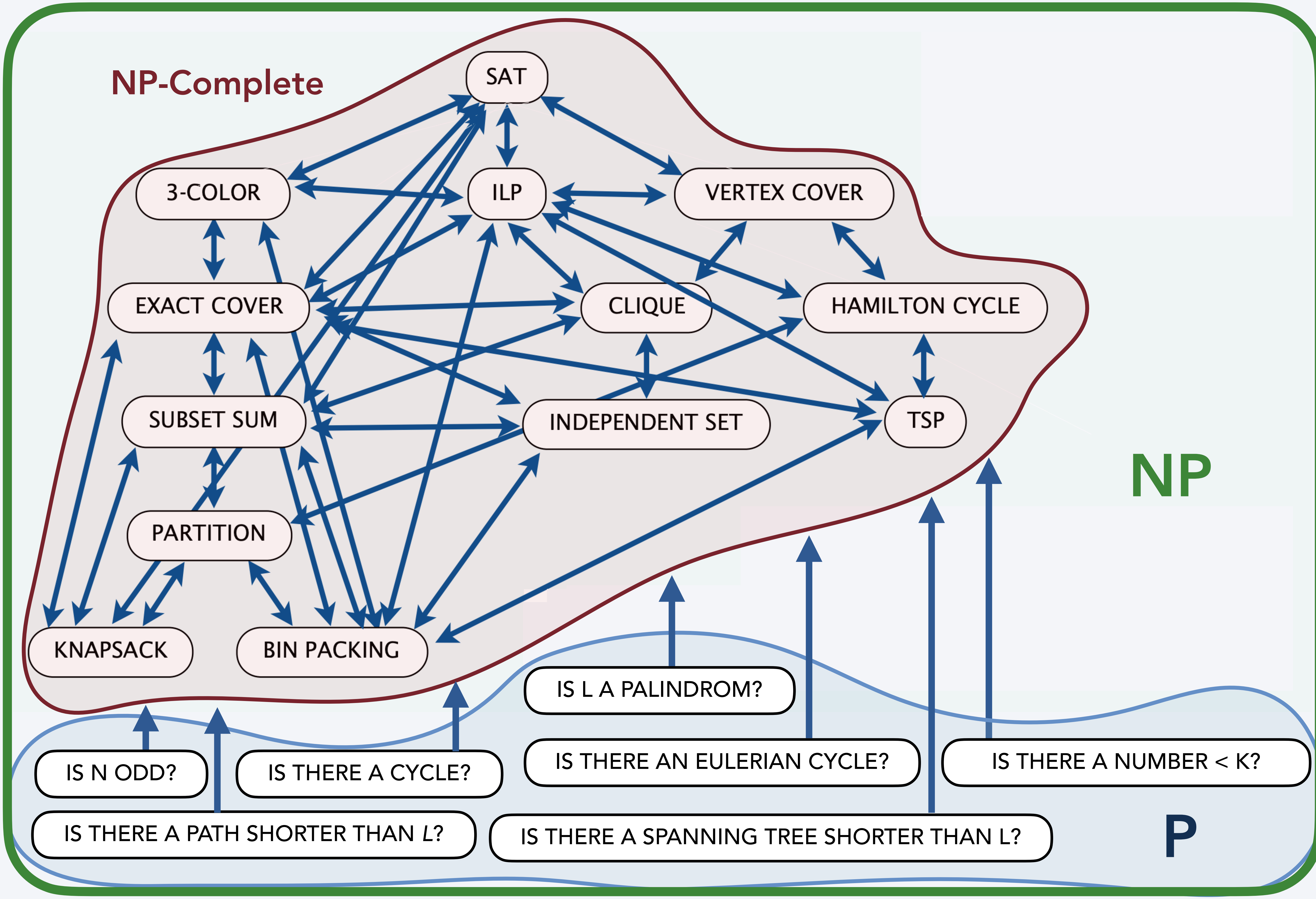



Steve Cook    Leonid Levin    Dick Karp

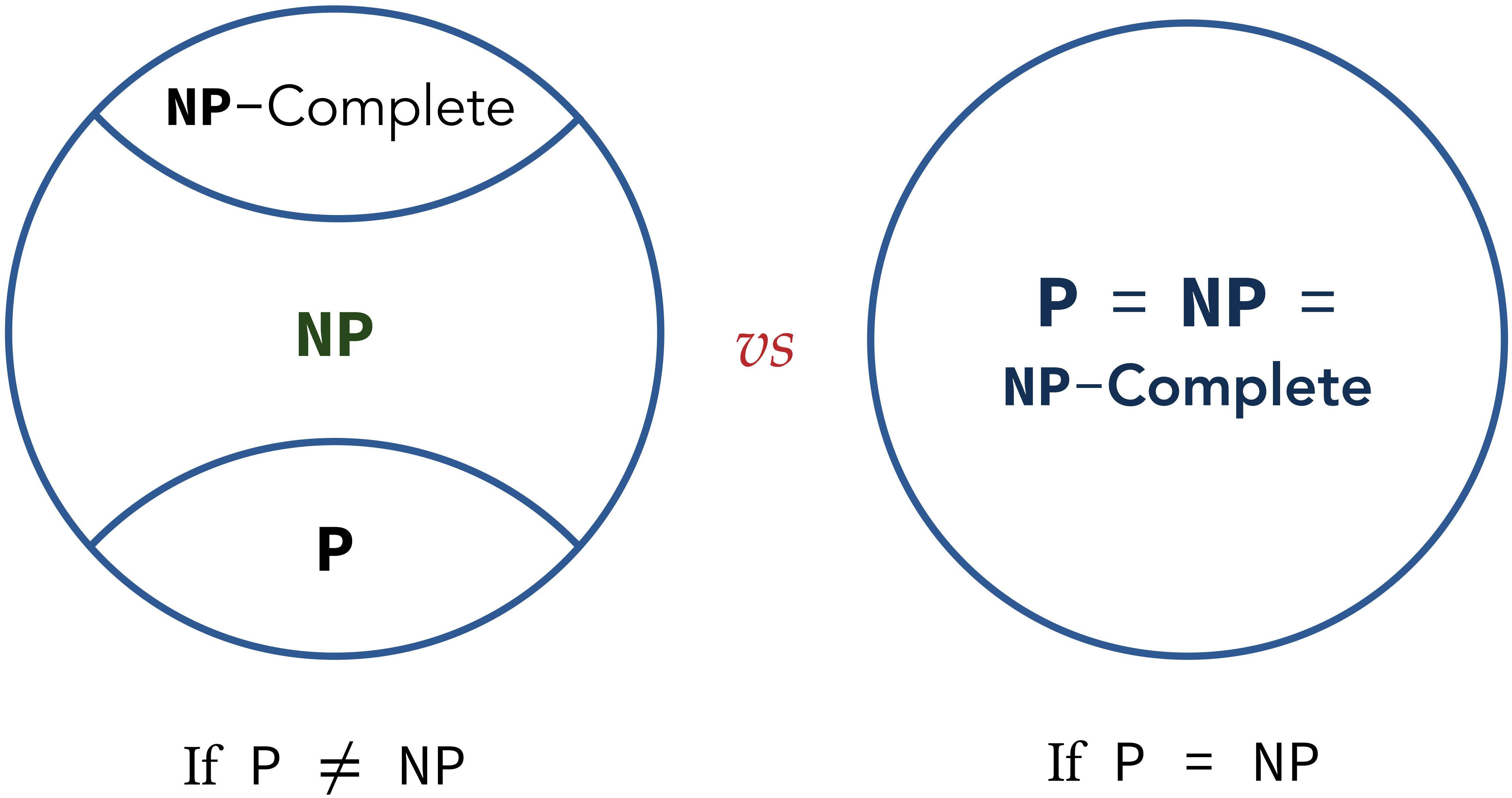
*All* of these problems are NP-complete.

A provably efficient algorithm for *any one* of them would yield a provably efficient algorithm for *all* of them

# World View if P != NP



# Again … Two Possible World Views



# NP-Completeness (Proof Examples)

`ILP` (*binary* Integer Linear Programming)

Given a system of inequalities, find a `0-1` solution.

$$
\begin{array}{rrrrrrl}
 & & x_1 & + & x_2 & \geq & 1 \\
x_0 & & & + & x_2 & \geq & 1 \\
x_0 & + & x_1 & + & x_2 & \leq & 2
\end{array}
$$

**Task**. Show that `ILP` is `NP`-Complete.

**Example**. A solution for the above is:

$$x_0 = 1, \quad x_1 = 1, \quad x_2 = 0$$

*Examples by Kevin Wayne*

# NP-Completeness (Proof Examples)

**ILP** (*binary* Integer Linear Programming)

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\begin{aligned}
x_1 + x_2 &\geq 1 \\
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x_0 + x_1 + x_2 &\leq 2
\end{aligned}
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**Example**. A solution for the above is:

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**Task**. Show that ILP is NP-Complete.

1. ILP is in NP.

Given values for the variables, we can verify in polynomial time if the inequalities are true.

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**Task**. Show that `ILP` is `NP`-Complete.

1. `ILP` is in `NP`.

Given values for the variables, we can verify in polynomial time if the inequalities are true.

2. `SAT` poly-time reduces to `ILP`.

$$
\begin{aligned}
\bar{x}_1 \lor x_2 \lor x_3 &= \text{TRUE} \\
x_1 \lor \bar{x}_2 \lor x_3 &= \text{TRUE} \\
\bar{x}_1 \lor \bar{x}_2 \lor \bar{x}_3 &= \text{TRUE} \\
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\bar{x}_2 \lor x_3 \lor x_4 &= \text{TRUE}
\end{aligned}
$$

**Example** SAT instance

$$
\begin{aligned}
(1 - x_1) + x_2 + x_3 &\geq 1 \\
x_1 + (1 - x_2) + x_3 &\geq 1 \\
(1 - x_1) + (1 - x_2) + (1 - x_3) &\geq 1 \\
(1 - x_1) + (1 - x_2) + x_4 &\geq 1 \\
(1 - x_2) + x_3 + x_4 &\geq 1
\end{aligned}
$$

**Equivalent** `ILP` instance.

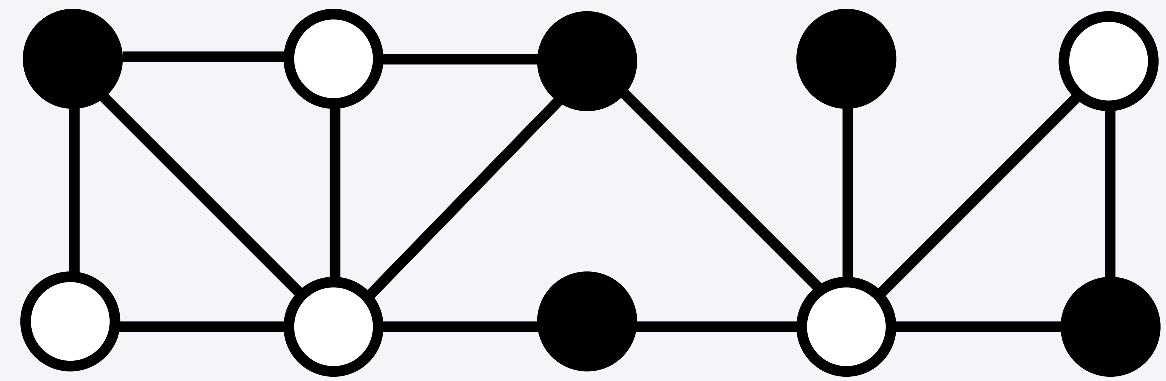
*Examples by Kevin Wayne*

# NP-Completeness (Proof Examples)



`INDEPENDENT-SET (IS)`

Given a graph and an integer *k*, is there a subset of *k* vertices such that no two vertices are adjacent?



**Example**. Black vertices form an independent set of size 5

**Task**. Show that `IS` is `NP`-Complete.

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optional

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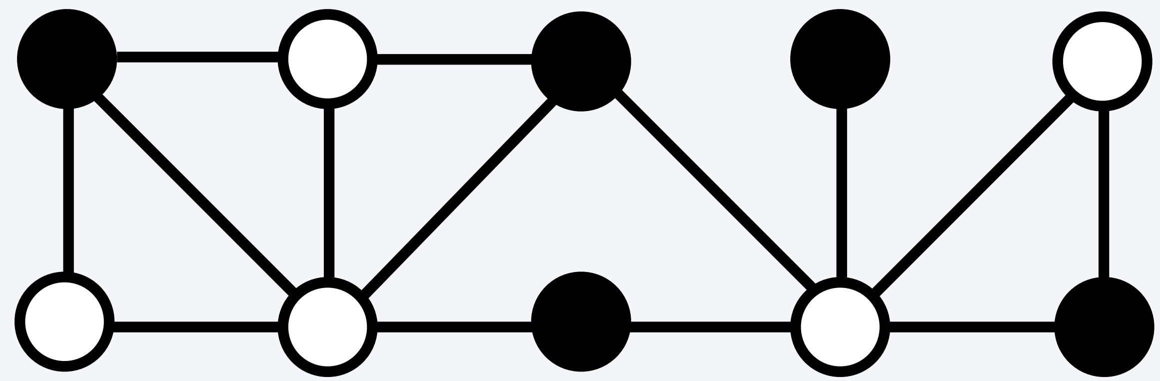

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1. IS is in NP.

Given a set $S$ of vertices in $G$, we can verify in polynomial time if any two are adjacent and if $|S| = k$.

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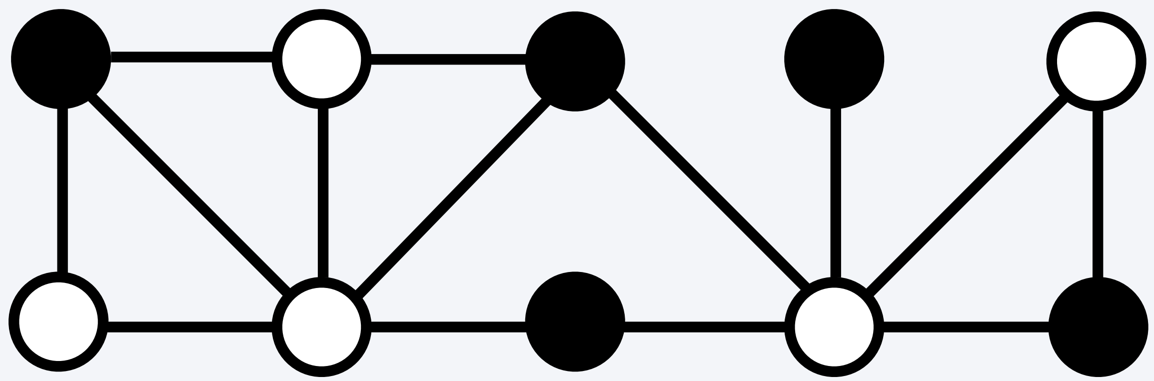

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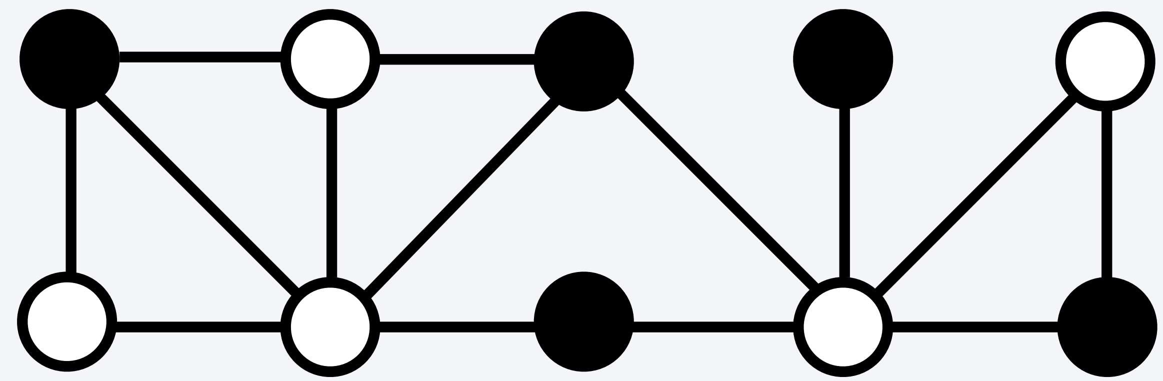

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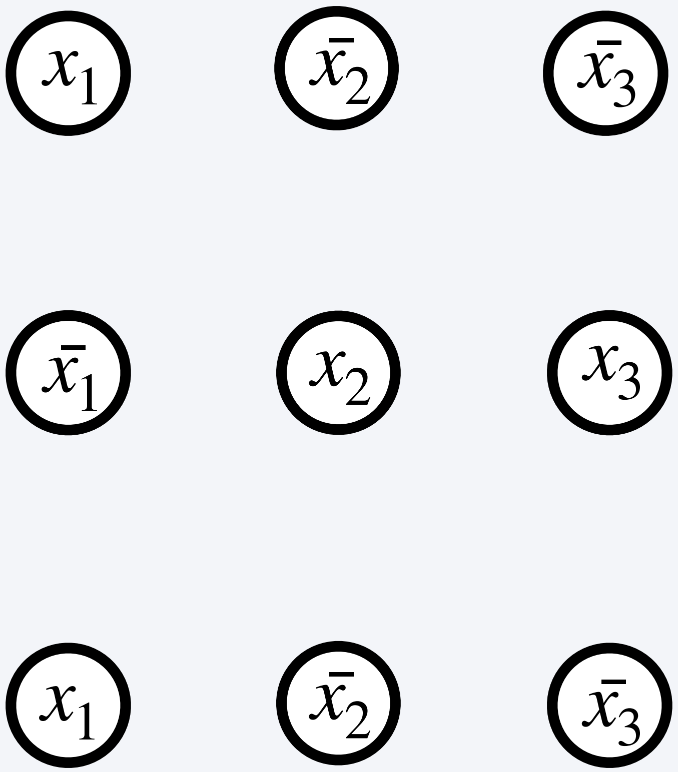
- Create a node for each literal in each clause.

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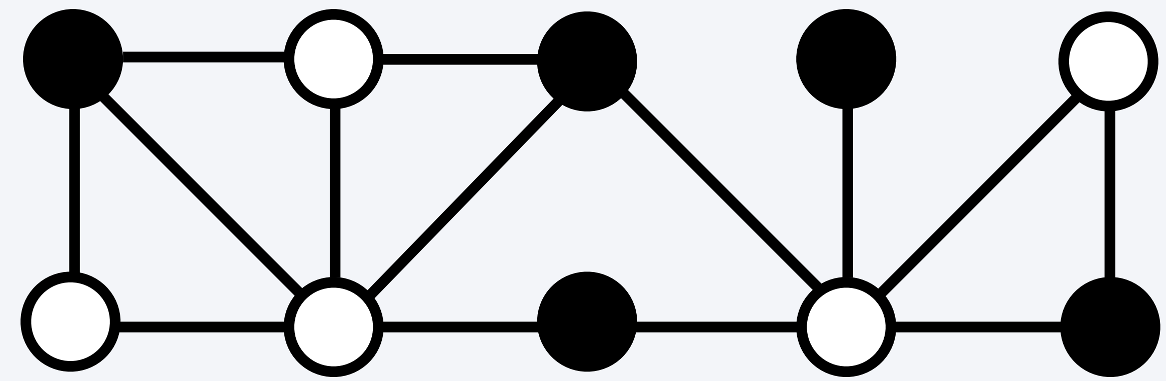

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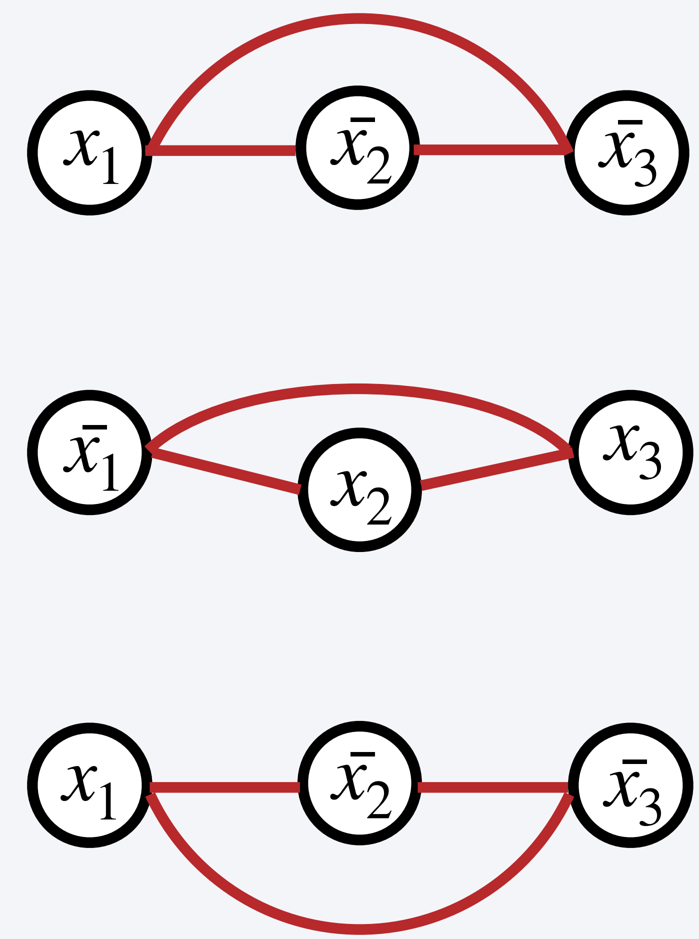


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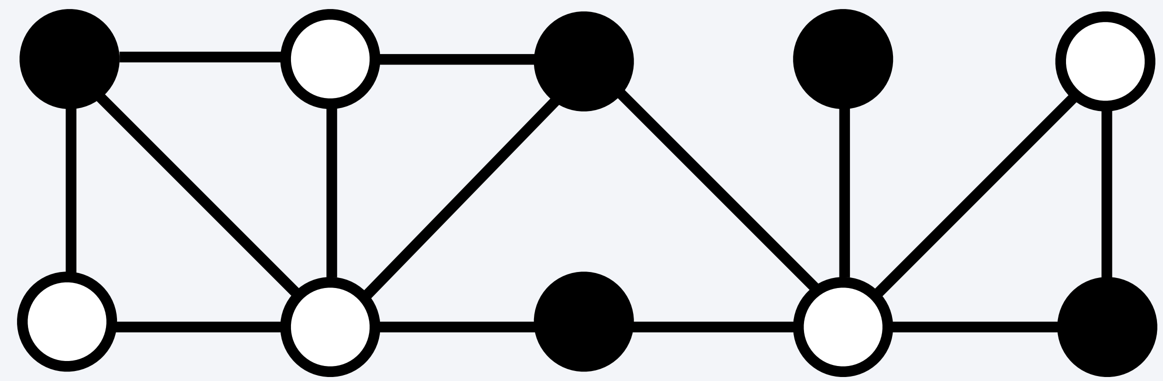

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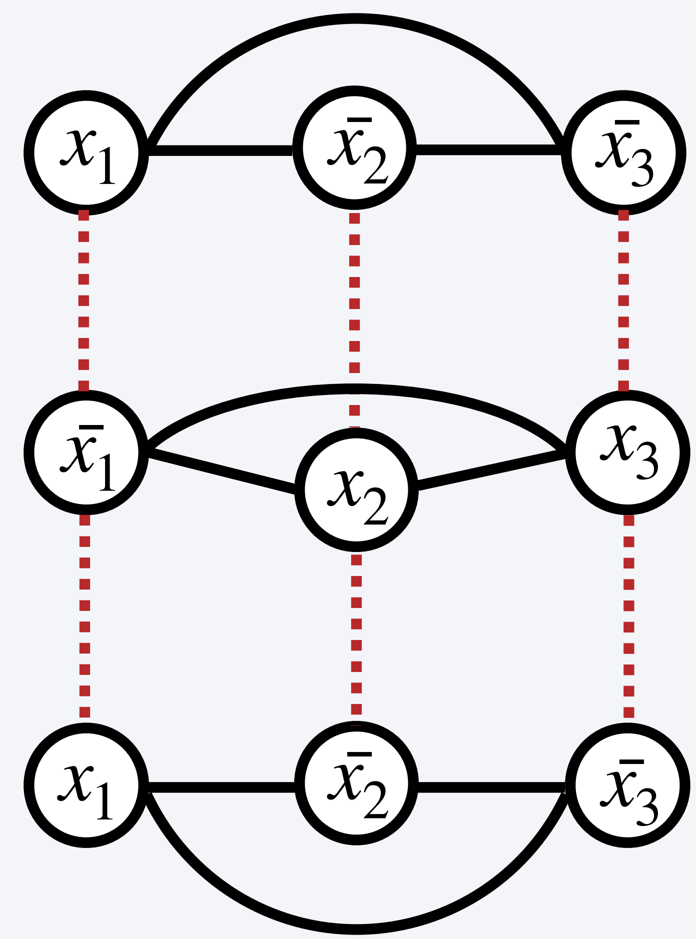


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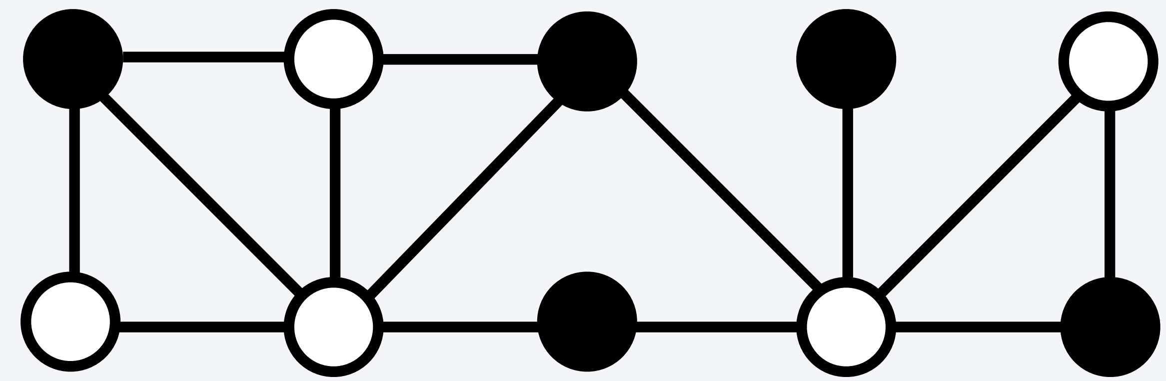

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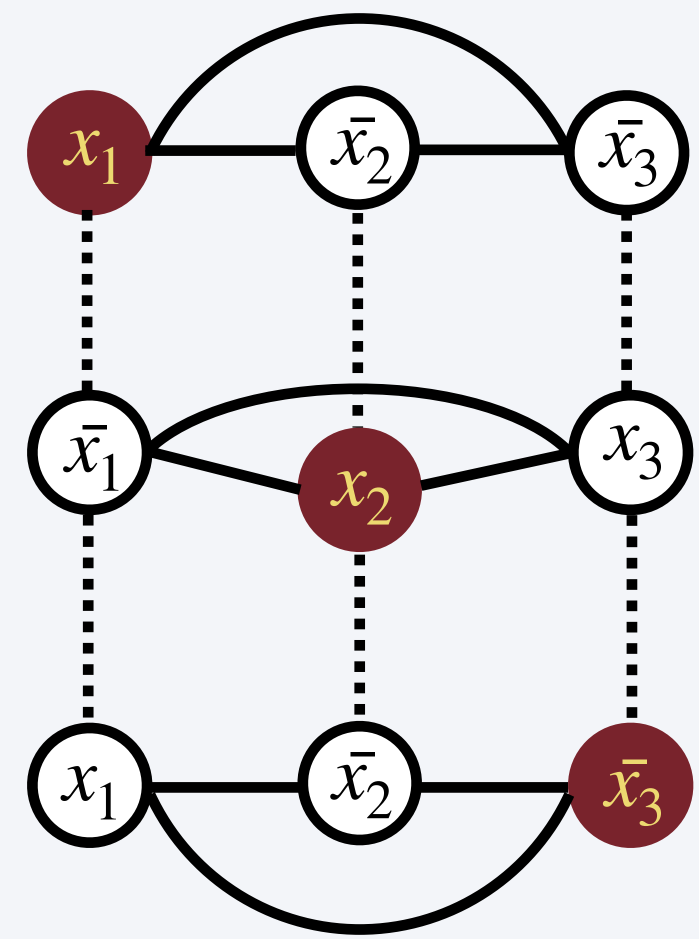
- Create a node for each literal in each clause.
- Connect each node to the literals in the same clause.
- Connect each literal to its negation.
- The expression is satisfiable iff there is an independent set of size = the number of clauses.



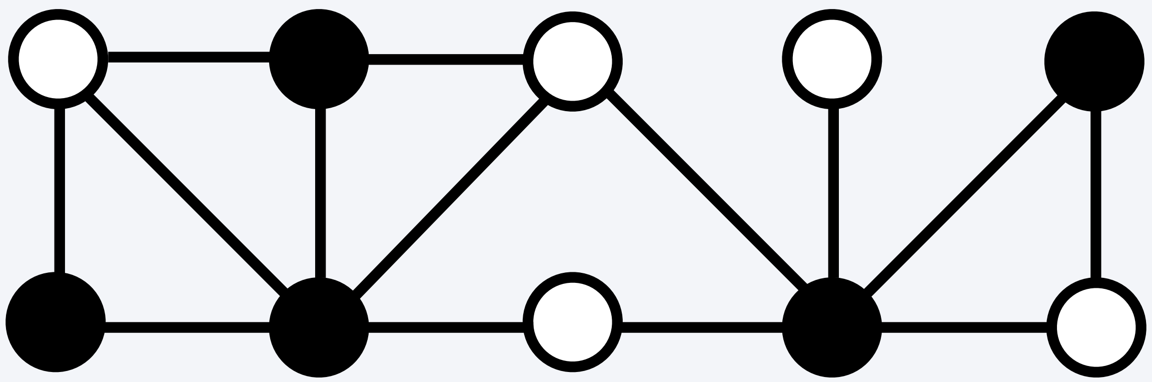

**Example.** $(x_1 \vee \bar{x}_2 \vee \bar{x}_3) \wedge (\bar{x}_1 \vee x_2 \vee x_3) \wedge (x_1 \vee \bar{x}_2 \vee x_3)$

# NP-Completeness (Proof Examples)



`VERTEX-COVER (VC)`

Given a graph and an integer *k*, is there a subset of *k* vertices such that each edge is incident to at least one vertex in the subset?



**Example**. Black vertices form a vertex cover of size 5

**Task**. Show that `VC` is `NP`-Complete.

# NP-Completeness (Proof Examples)



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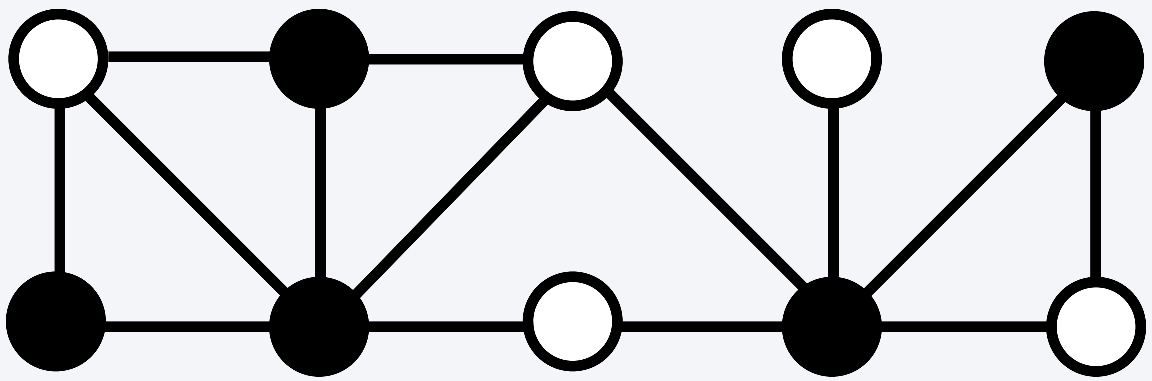

**Example**. Black vertices form a vertex cover of size 5

**Task**. Show that VC is NP-Complete.

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Given a set $S$ of vertices in $G$, we can verify in polynomial time if each edge in the graph is incident to a vertex in $S$ and if $|S| = k$.

# NP-Completeness (Proof Examples)

optional

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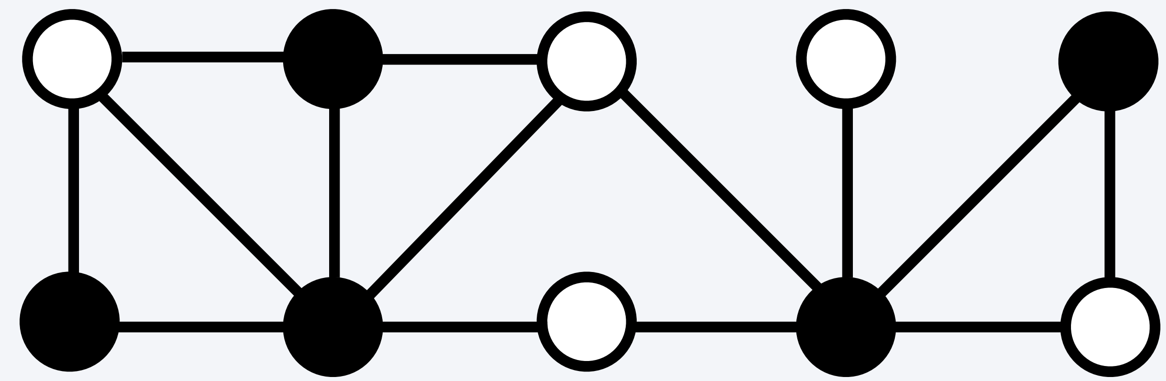

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2. INDEPENDENT-SET poly-time reduces to VERTEX-COVER.

We can pick *any* **NP**-Complete problem for the reduction, not necessarily **SAT**!

# NP-Completeness (Proof Examples)

optional

96

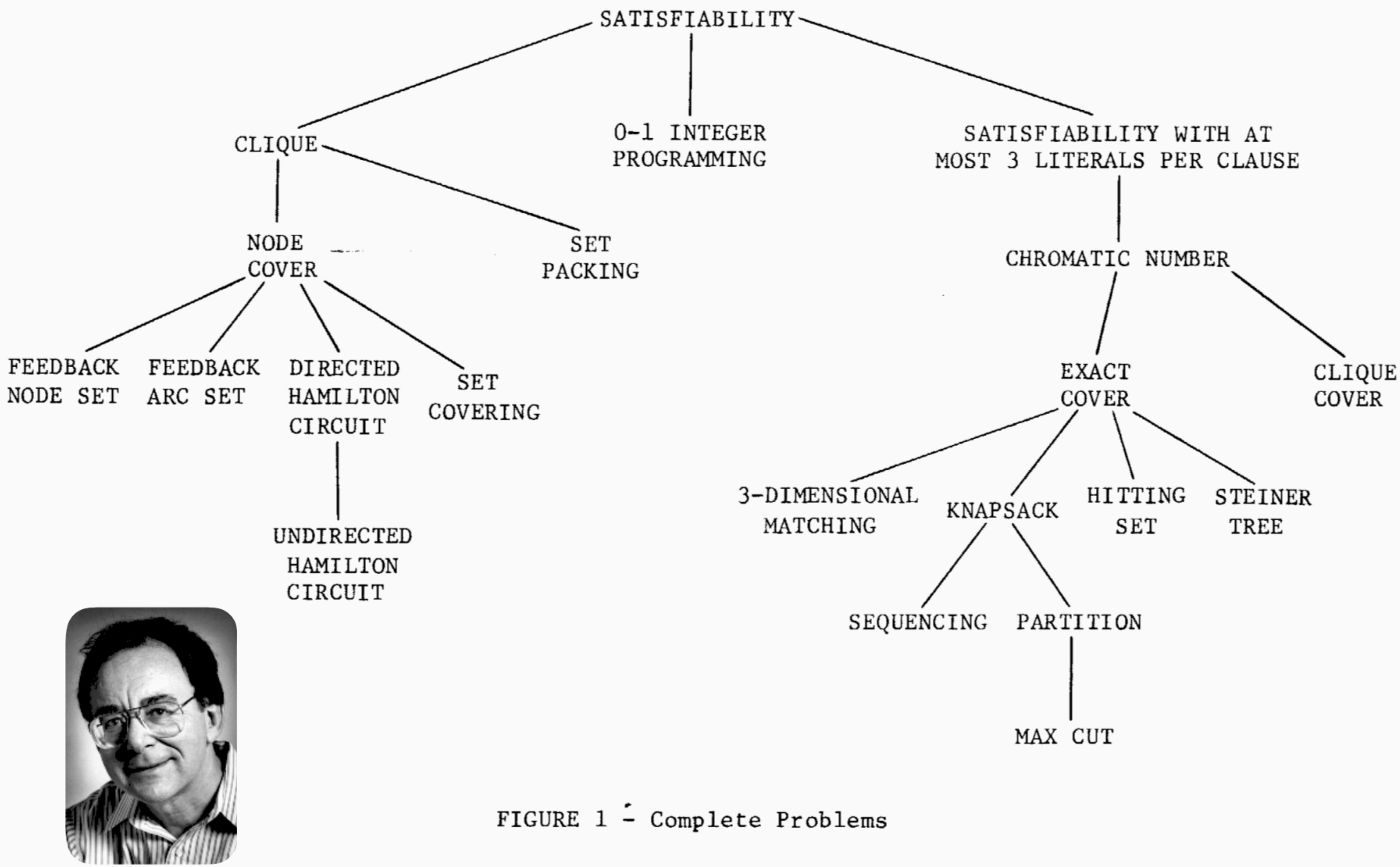


FIGURE 1 - Complete Problems

RICHARD M. KARP

**Dick Karp (1972)**
**1985 Turing Award**

# NP-Completeness (Proof Examples)

optional

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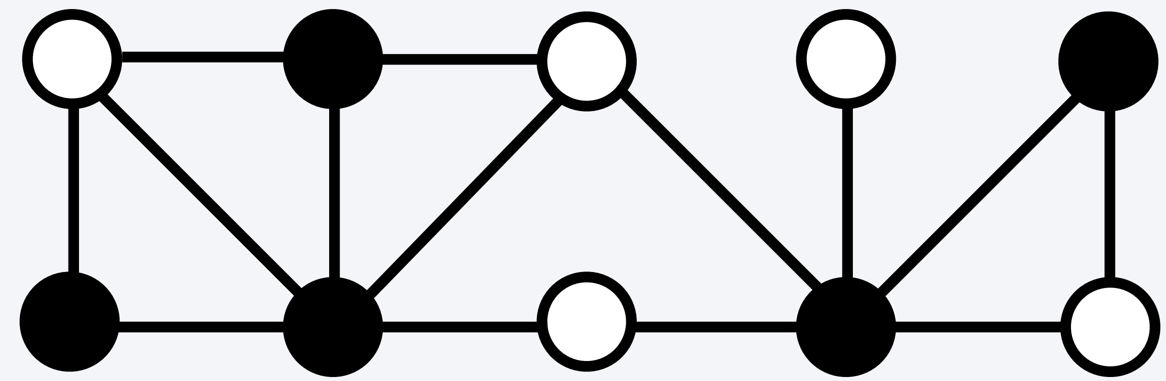

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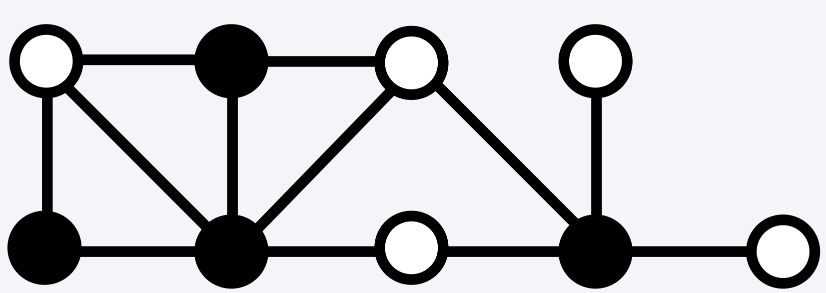

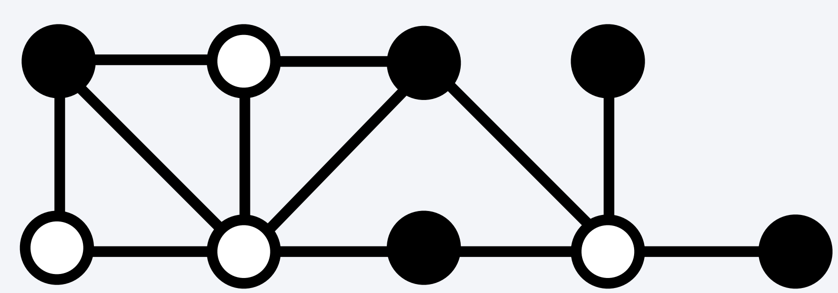

# NP-Completeness (Proof Examples)

optional

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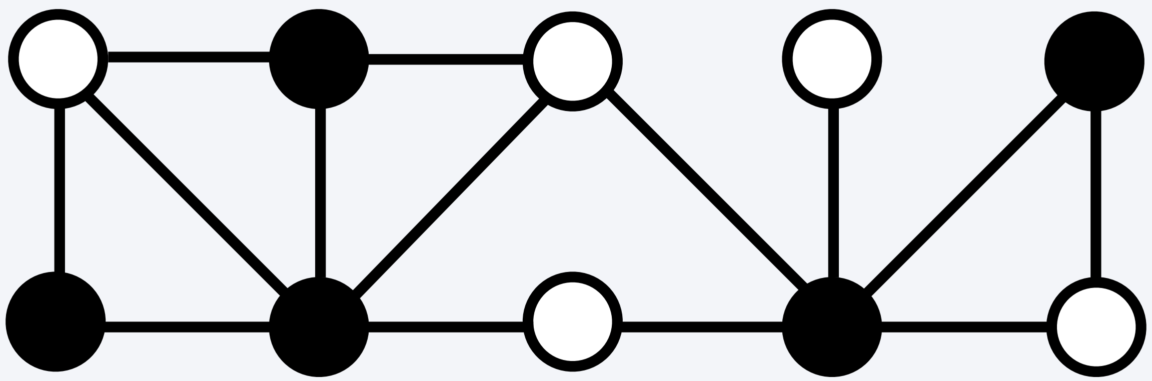

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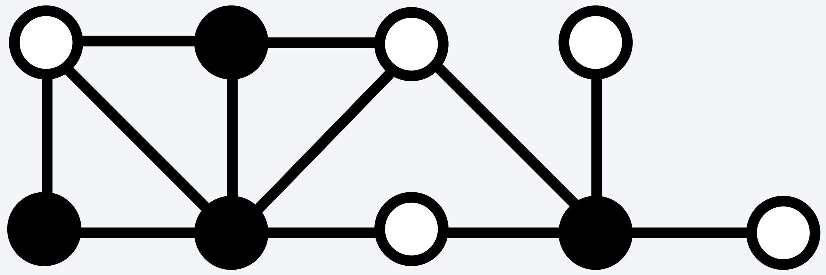
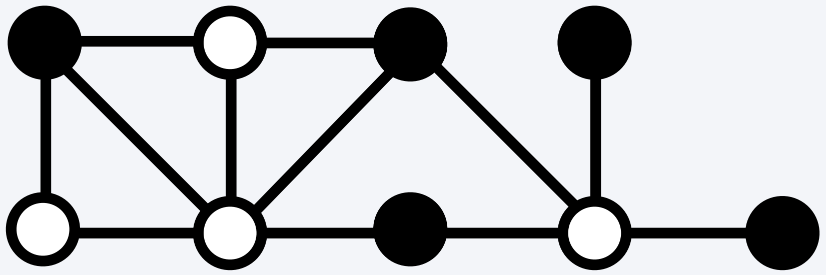
$S$ is an independent set of size $k$ iff $V - S$ is a vertex cover of size $n - k$.



Vertex Cover of size **4**



Independent Set of size **5**

# NP-Completeness (Proof Examples)

TRAVELING SALESMAN PROBLEM (TSP)

Given a complete weighted graph $G$, does $G$ contain a simple circuit $C$ that visits each node exactly once of *length* $\leq T$?

**Task**. Show that TSP is NP-Complete.

1. Show that TSP is in NP. ← straight-forward

# NP-Completeness (Proof Examples)

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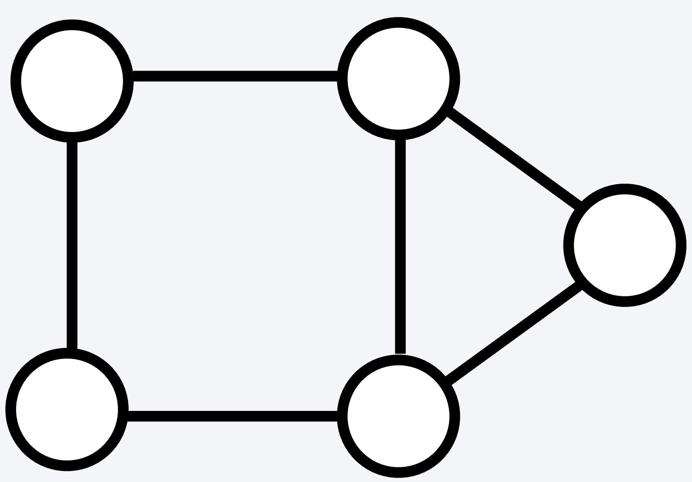
**Task**. Show that TSP is NP-Complete.

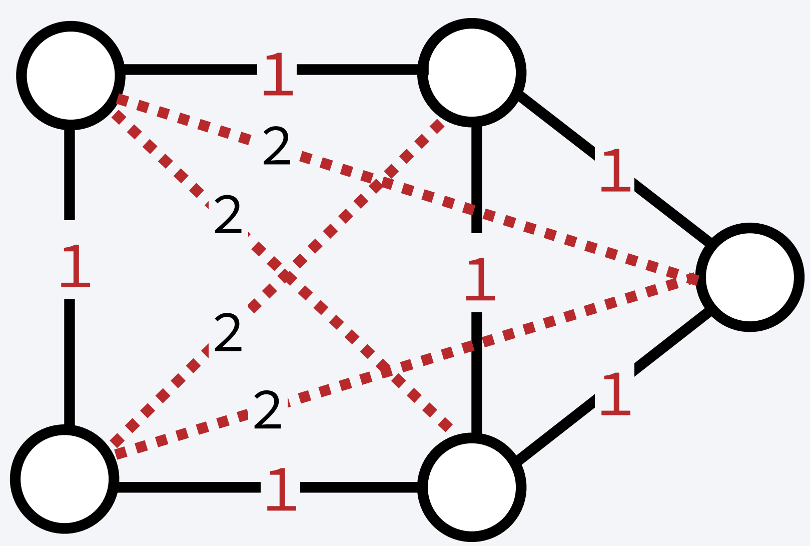
1. Show that TSP is in NP. ← straight-forward

2. HAMILTONIAN poly-time reduces to TSP.



$G$

Input to the HAMILTONIAN




$G'$

Input to TSP

$G$ has a hamiltonian cycle iff $G'$ has a tour of length $V$

Add edge $(u, v)$ with weight **1** if $(u, v)$ is in $G$.
Add edge $(u, v)$ with weight **2** if $(u, v)$ is not in $G$.

# Quiz # 10

Are there problems that are in **NP** but are not in **P** and are not **NP**-Complete.

**A.** Yes.

**B.** No.

**C.** None of the above.

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Yes if $\mathbf{P} \neq \mathbf{NP}$.

No if $\mathbf{P} = \mathbf{NP}$.

# Quiz # 10

Are there problems that are in **NP** but are not in **P** and are not **NP**-Complete.

**A.** Yes.

**B.** No.

**C.** None of the above.

Yes if $\mathbf{P} \neq \mathbf{NP}$.

No if $\mathbf{P} = \mathbf{NP}$.

There are, however, problems in NP that we could not yet prove to be in P and could not also prove to be NP-Complete!

**Examples.** Integer Factoring and Graph Isomorphism.

# Definitions (Complexity Classes)

### Class **P**.

A decision problem is in P if it is solvable in polynomial time
(i.e. in $O(n^c)$, where $n$ is the input size and $c$ is a constant)

### Class **NP**.

A decision problem is in NP if it is verifiable in polynomial time.
(Given an instance $I$ or a problem $P$ and a witness $W$ for the solution, can we verify in polynomial time if $W$ proves that the answer for $I$ is yes?)

### Class **NP−Complete**.

A decision problem is NP−Complete if:

- It is in NP.
- All problems in NP reduce to it in polynomial time.

### Class **NP−Hard**.

A problem is NP−Hard if all problems in NP reduce to it in polynomial time.
(*at least as hard as the hardest problems in* NP)

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Examples.

- All NP-Complete Problems.
- TSP Optimization.
- Finding the Longest Simple Path.

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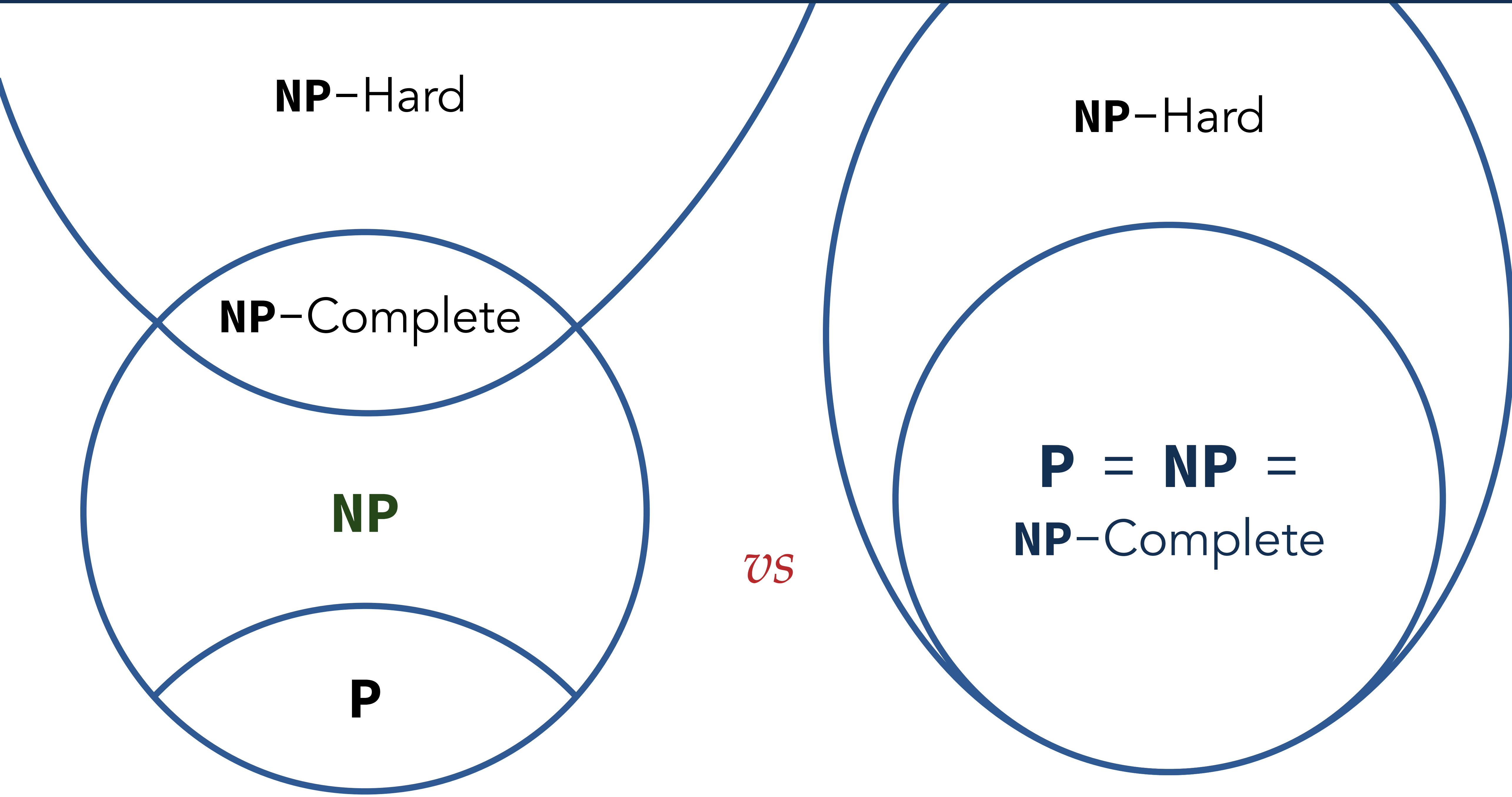
# Two Possible World Views




If $P \neq NP$

If $P = NP$

# Living with Intractability

When you encounter an NP-complete problem

- It is safe to assume that it is intractable.
- What to do?

does not have an algorithm that solve all instances in polynomial time.

Four successful approaches

- Don't try to solve intractable problems.
- Try to solve real-world problem instances.
- Look for approximate solutions (not discussed in this lecture).
- Exploit intractability.

*slide by Kevin Wayne*

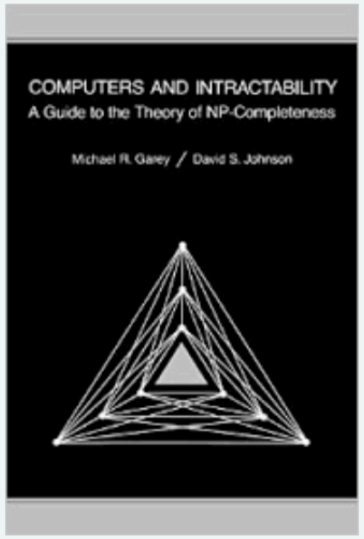
# Living with Intractability: Don't Try To Solve It!

**Knows no theory**

*I can't find an efficient algorithm.*
*I guess I'm just to dumb.*

**Knows computability**

*I can't find an efficient algorithm,*
*because no such algorithm is possible!*




**Knows intractability**

*I can't find an efficient algorithm,*
*but neither can all these famous people!*

*slide by Kevin Wayne*

# Living with Intractability: Solve Real-World Instances

Observations

- Worst-case inputs may not occur for practical problems.
- Instances that do occur in practice may be easier to solve.

Reasonable approach: relax the condition of *guaranteed* poly-time algorithms.

SAT

- *Chaff* solves real-world instances with 10,000+ variables.
- Princeton senior independent work (!) in 2000.

TSP

- *Concorde* routinely solves large real-world instances.
- 85,900-city instance solved in 2006.

ILP

- *CPLEX* routinely solves large real-world instances.
- Routinely used in scientific and commercial applications.

TSP solution for 13,509 US cities

*slide by Kevin Wayne*