CS11313 - **Spring** 2022

Design & Analysis of Algorithms

Greedy Algorithms

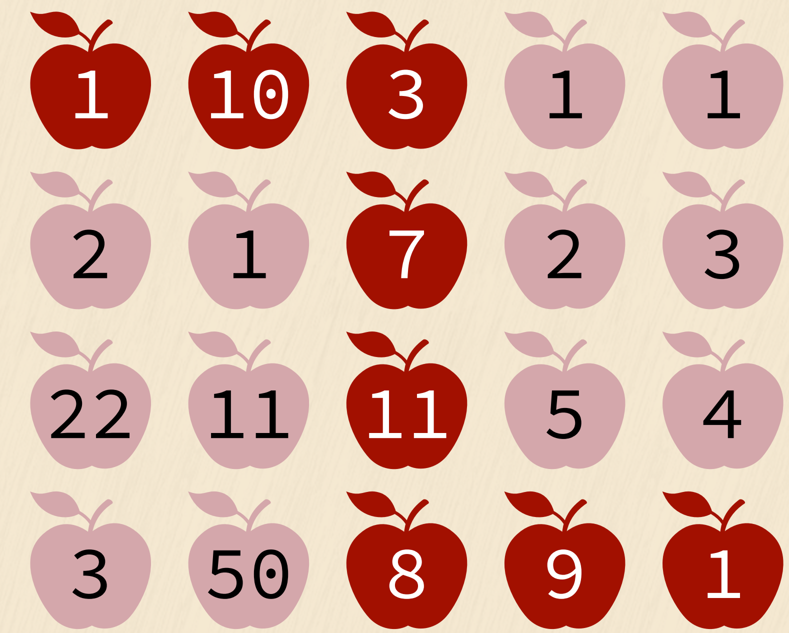
Ibrahim Albluwi

Reminder: Collecting Apples

Problem Description.

- **Goal.** Collect as many apples as possible.
- **Constraints**. Move *right* or *down* only.
- **Input**. The matrix `apples[N][M]`
  `apples[i][j]` is the number of apples at cell `[i][j]`.

start



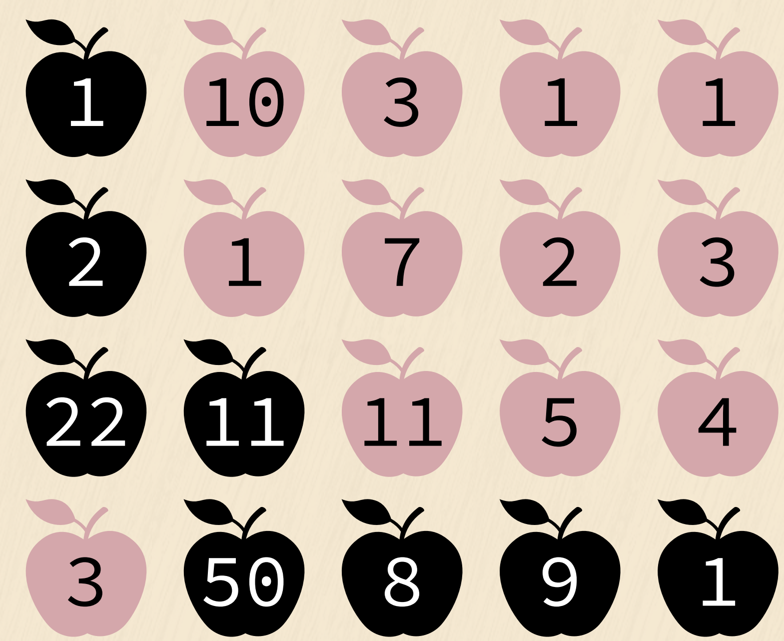
end

Total = 50

Solution # 1.

```
if apples[i+1][j] > apples[i][j+1]:
      go down.
else go right.
```

FAIL



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A **GREEDY** efficient solution that does not always work!

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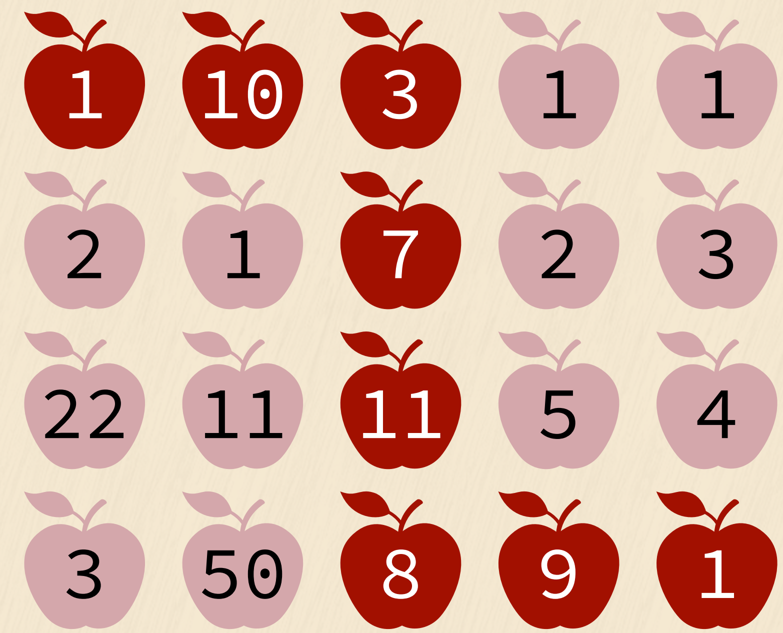

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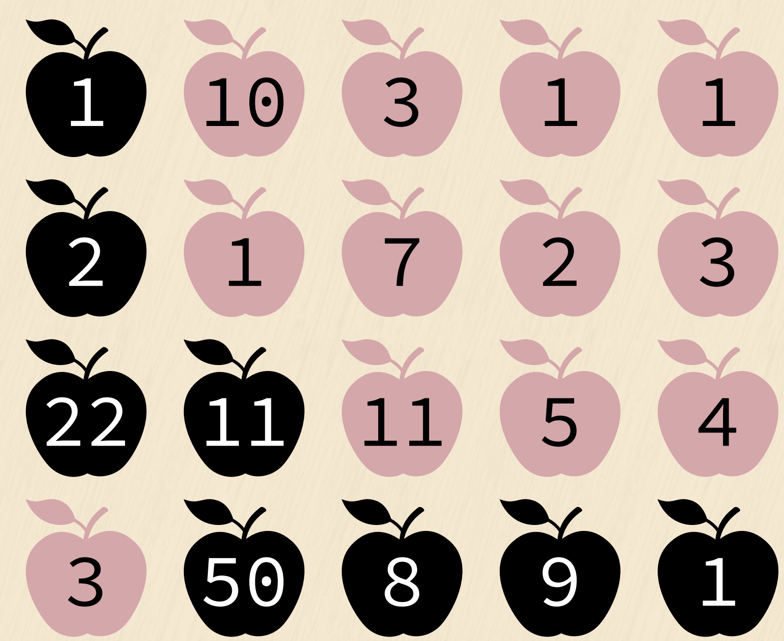

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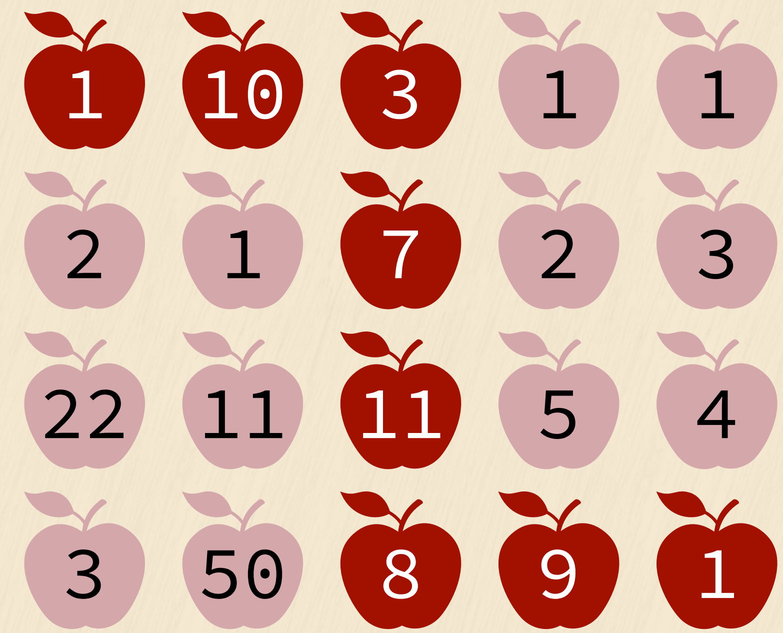

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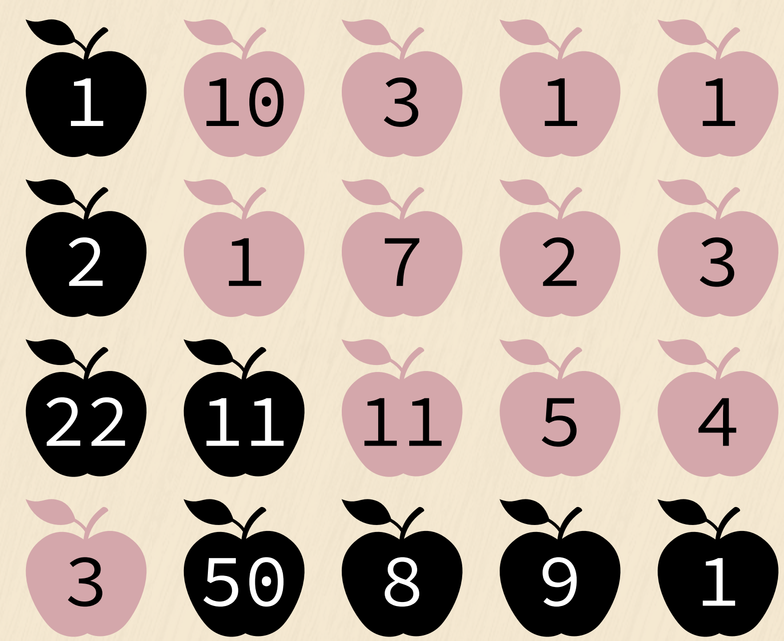
Which problems have greedy solutions that always work?



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Greedy Algorithms

Optimization problem. The problem of finding the *best* solution among all *feasible* solutions.

maximizes or
minimizes an
objective function

valid
(*respects the constraints*)

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Objective Function: Choose the subset of items with the *maximum* value.
Constraints: The *total weight* of the chosen items must be *less* than the knapsack capacity.

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Greedy 1. Take it if it is the *most valuable* item!

Greedy 2. Take it if it is the *lightest* item!

Greedy 3. Take it if it has the *maximum value per Kg*.

Greedy Algorithms vs Dynamic Programming

Similarities. Both can be used to solve *optimization problems* and both work on problems with an *optimal substructure*.

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Example. **Knapsack**(Item 1 | Item 2 | Item 3 | Item 4 | Item 5 ······ Item *n* | W)

Greedy. Decide (without much effort) whether to pick Item 1 or not and solve: `Knapsack(2 ... n, W-`w_1`)` *or* `Knapsack(2 ... n, W)`

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We need a **proof** that such a decision is safe !!

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Dynamic Programming.

- Try picking Item 1 and solve **Knapsack**(2 ... n, $W-w_1$)
- Try not picking Item 1 and solve **Knapsack**(2 ... n, W)

Decide which is better.

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Greedy choice # 1. Take the *most valuable* item first.

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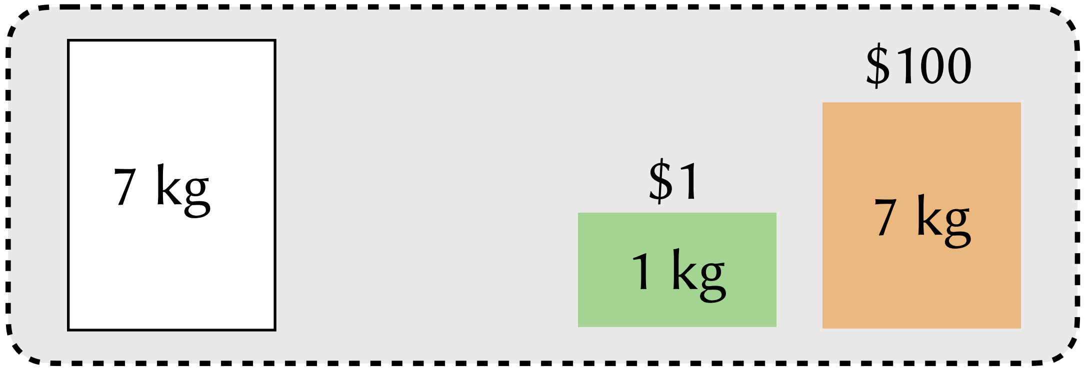

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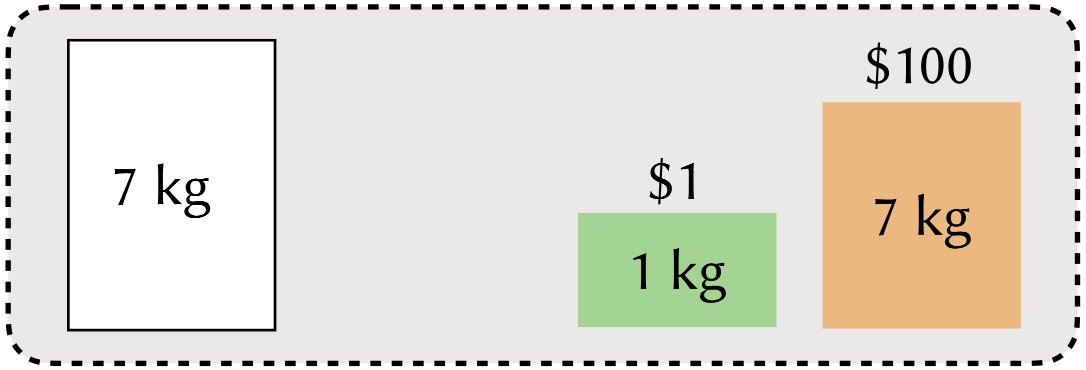

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Greedy choice # 3. Take the item with the *highest value per Kg*.

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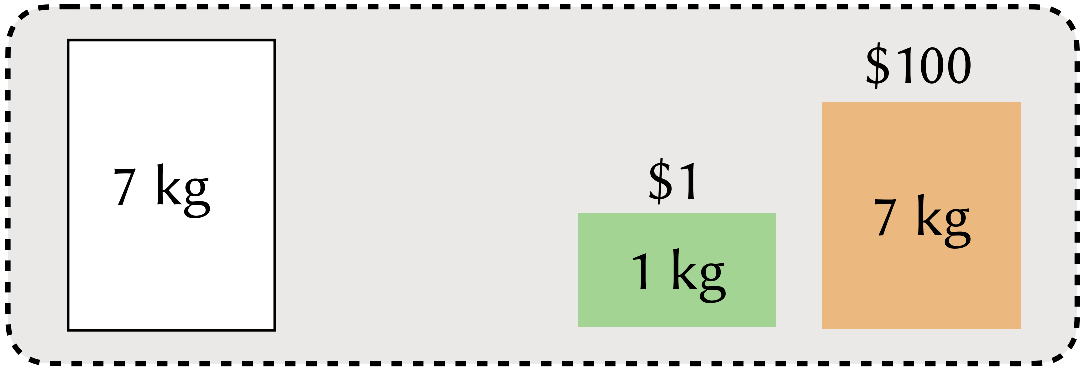

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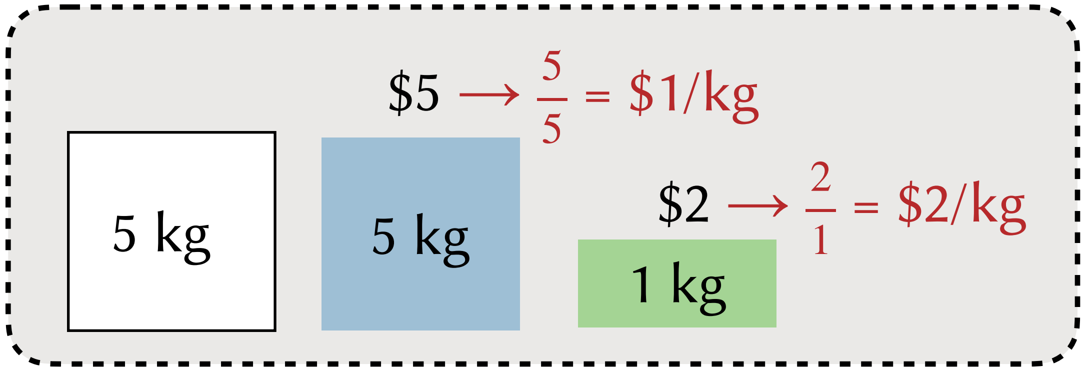

Counter Example. Greedy gives $2 but the optimal is $5.

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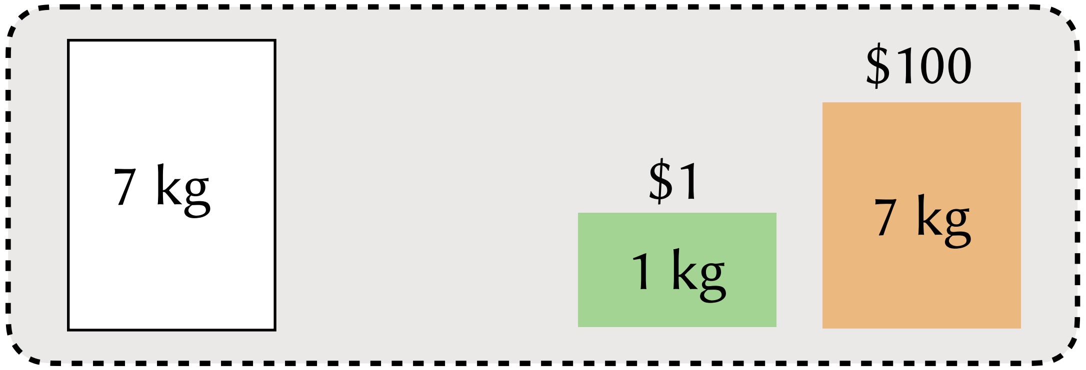

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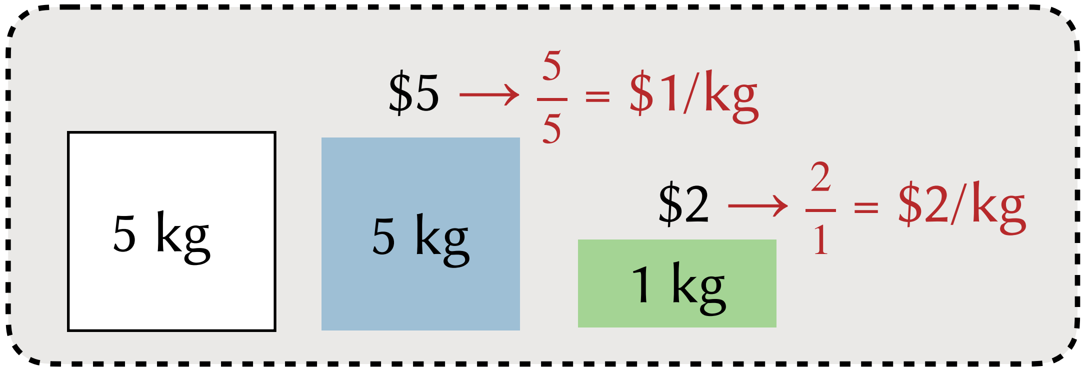

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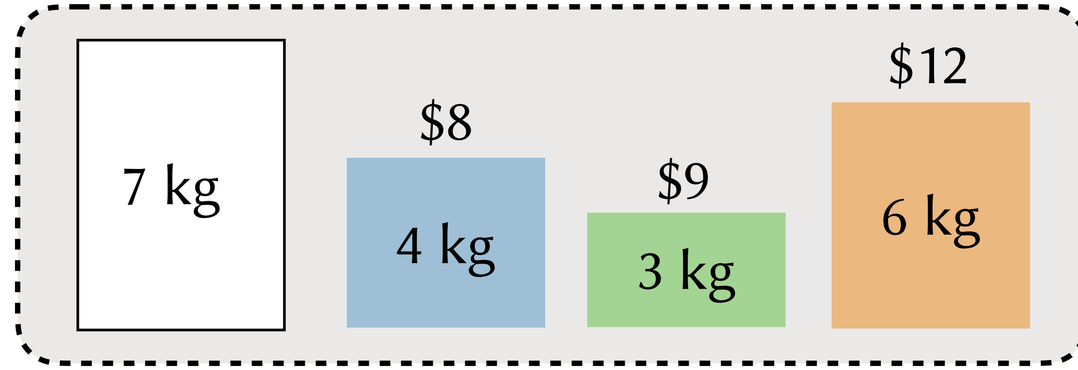

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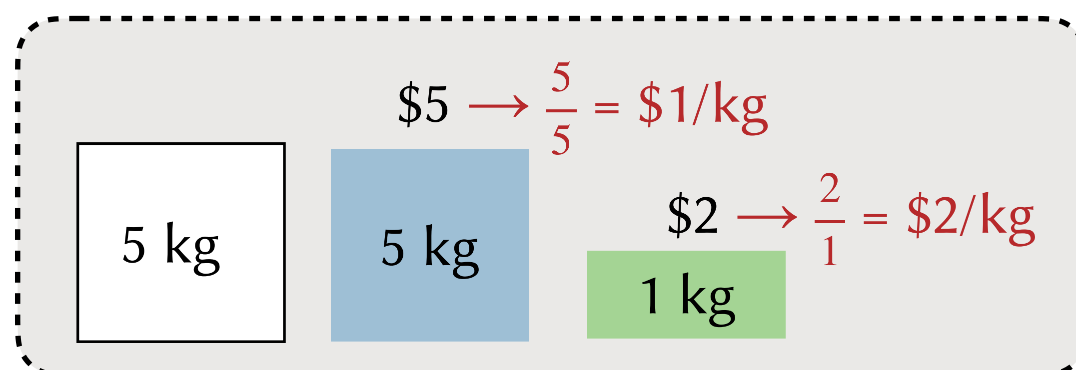

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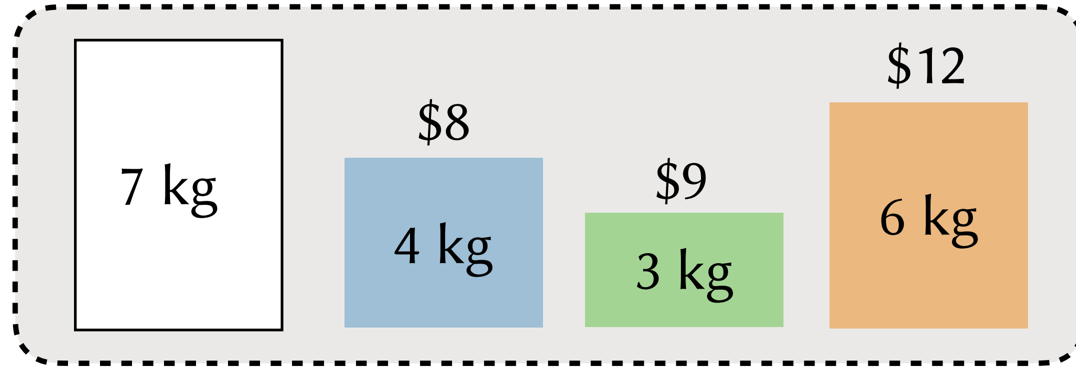
To prove that a greedy strategy is *not optimal* it is enough to find *one counter example*.

There is no known optimal greedy strategy for the 0-1 Knapsack problem



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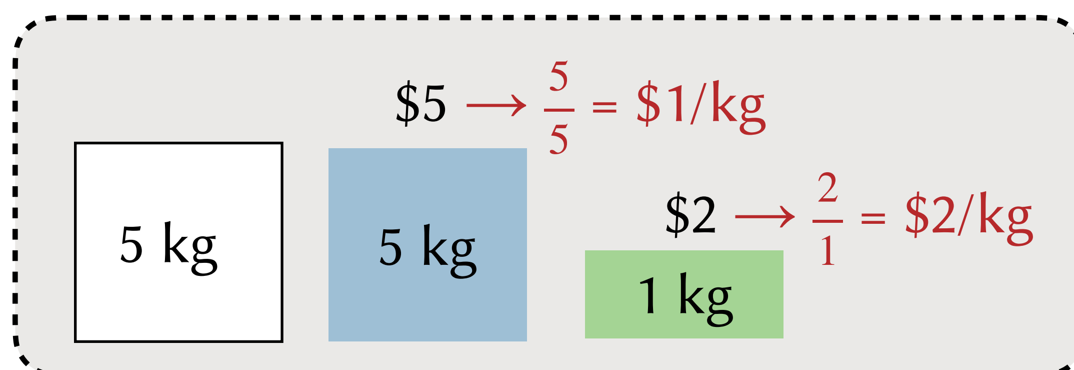

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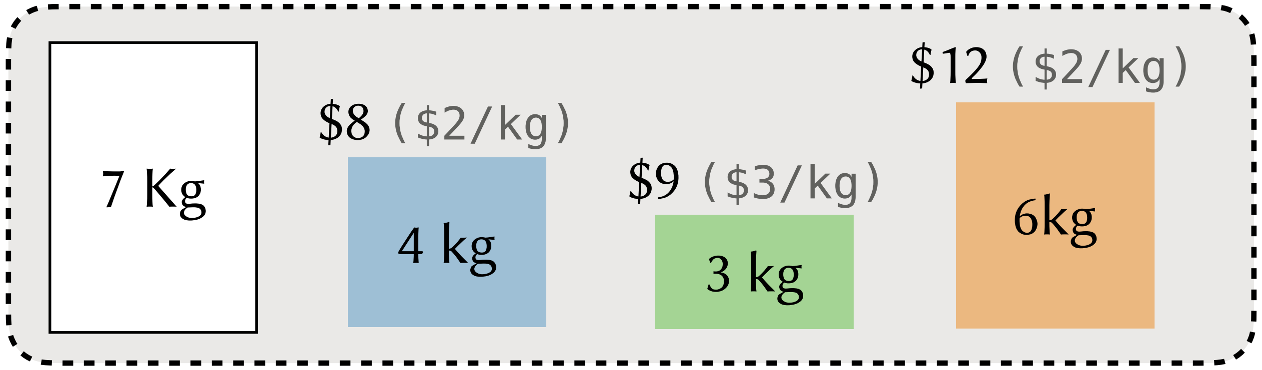
Greedy choice.
Start filling with the item that has *highest value per Kg*.



(3 x $3/kg) + (4 x $2/kg)
= $17

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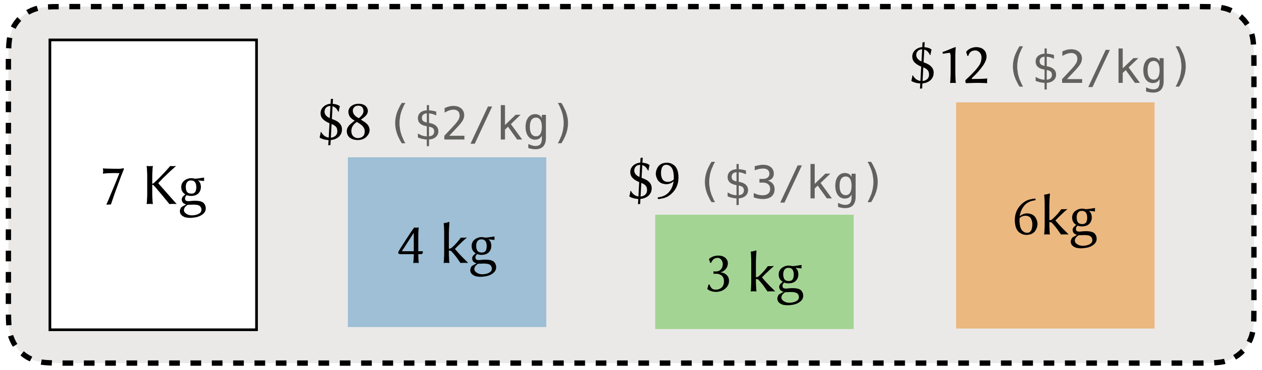

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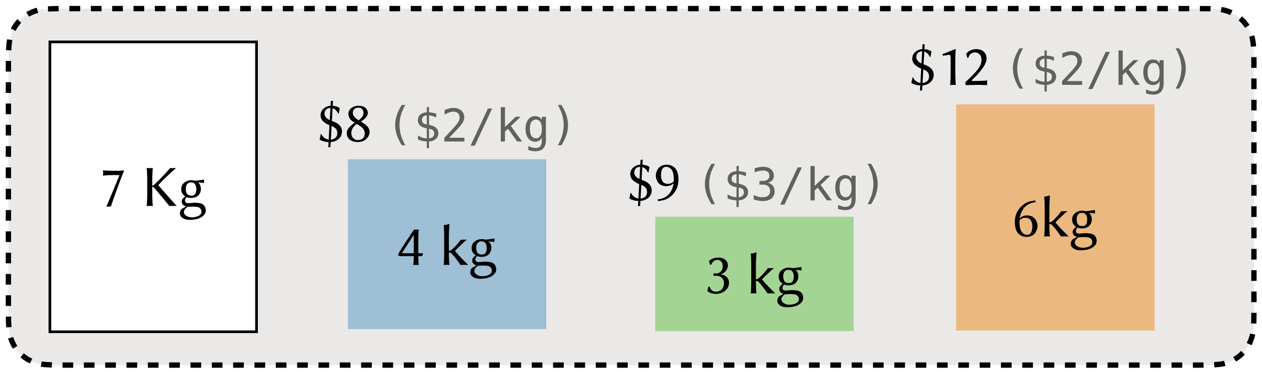

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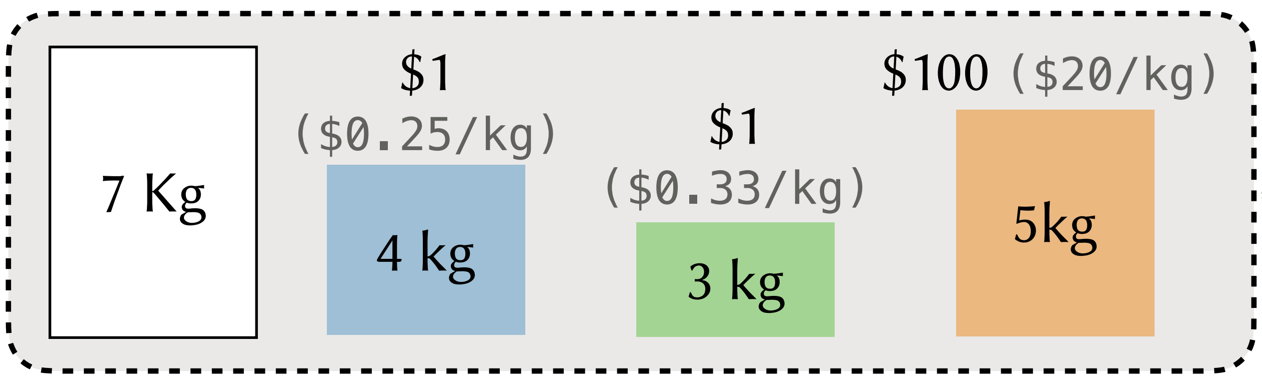

(5 x $20/kg) + (2 x $0.33/kg)
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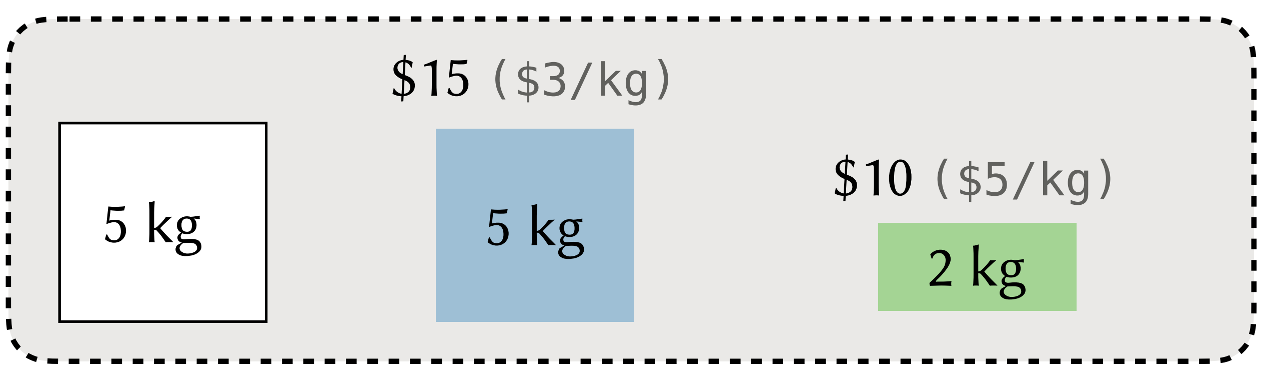

(2 x $5/kg) + (3 x $3/kg)
= $19

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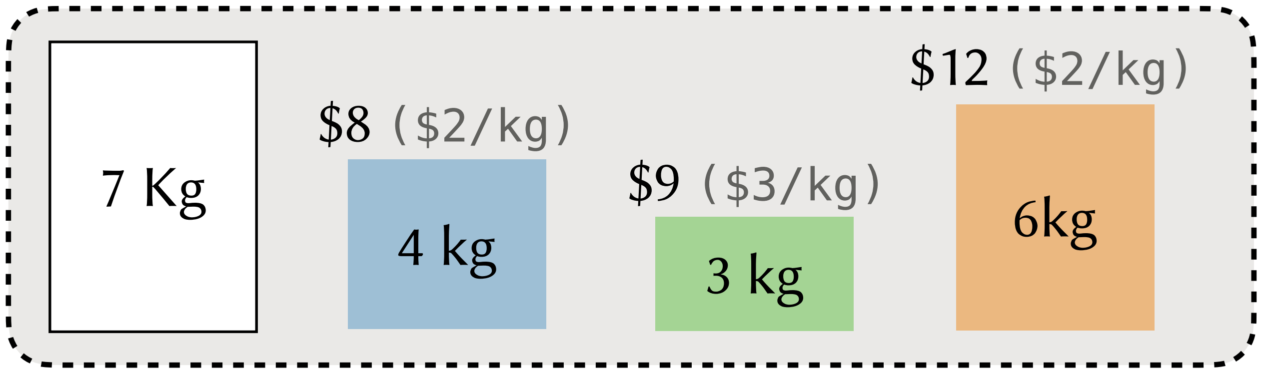
This algorithm is *optimal*.
However, we need a proof!



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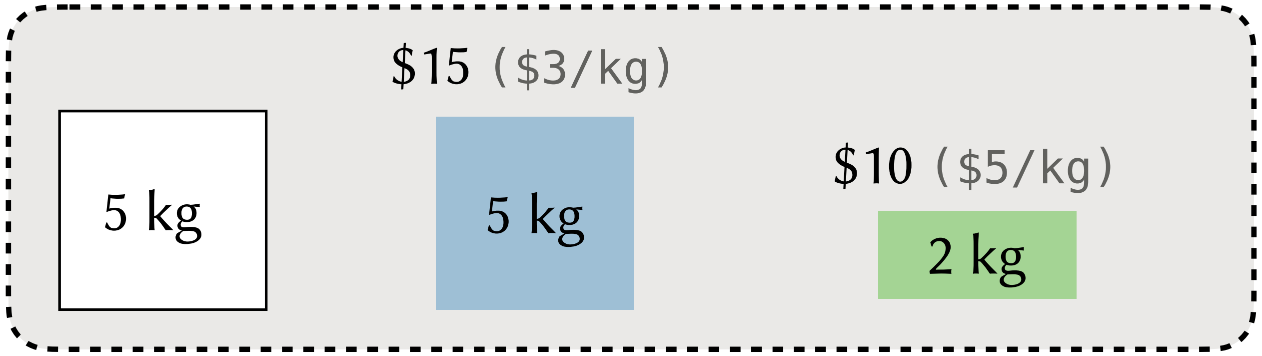

(2 x $5/kg) + (3 x $3/kg)
= $19

Fractional Knapsack

Consider a set of n items sorted by value per weight.

Item 1		Item 2		Item 3		Item 4		Item 5		········		Item n
$\frac{v_1}{w_1}$	$>$	$\frac{v_2}{w_2}$	$>$	$\frac{v_3}{w_3}$	$>$	$\frac{v_4}{w_4}$	$>$	$\frac{v_5}{w_5}$	$>$		$>$	$\frac{v_n}{w_n}$

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Assume that there is an optimal solution for the fractional knapsack problem that *does not consume the item with the highest value per Kg.*

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Assume that there is an optimal solution for the fractional knapsack problem that *does not consume the item with the highest value per Kg.*

We can replace parts of this item with parts of *item* 1 and get a better solution! Therefore, the solution is not optimal! (*contradiction*)

Note. This is not a proof. However, it captures the essence of the argument for why this greedy choice is optimal.

Fractional Knapsack (Algorithm)

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FRACTIONAL_KNAPSACK(w[], v[], n, W)

Sort w and v by v[i]/w[i] in decreasing order
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while i <= n and load < W:
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load = 0, i = 1
while i <= n and load < W:

    if w[i] <= W-load:
```

all of the item fits inside the remaining capacity

```
     else
```

only a fraction of the item fits inside the remaining capacity

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while i <= n and load < W:

   if w[i] <= W-load:
        Take all of item i
        load += w[i]
        profit = profit + v[i]
    else
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        Take (W - load) of item i
        profit = profit + (W - load) * v[i]/w[i]
        load = W

    i = i+1
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$\Theta(n \log n)$

$O(n)$

Running Time. $\Theta(n \log n)$

Lesson. Greedy algorithms are typically simple and efficient.

Change-Making Problem

Problem. Given a set of coin denominations (e.g. quarter, dime, nickel, etc.) and an amount of money, find the *minimum number of coins* that add up to this amount of money.

Example. What is the minimum number of coins needed for paying 99¢ using Jordanian currency?

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1x50¢ + 1x25¢ + 2x10¢ + 4x1¢

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Counter Example. Consider the denominations `{25¢, 20¢, 10¢, 1¢}`. What is the minimum number of coins needed to pay 40¢ ?

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Greedy: `25¢ + 10¢ + 1¢ + 1¢ + 1¢ + 1¢ + 1¢ = 40¢` (7 coins).
Optimal: `20¢ + 20¢ = 40¢` (2 coins)

Interview Problem

💪 optional

Problem. Given n coin denominations $\{ d_1, d_2, \ldots, d_n \}$ and a target value V, find the fewest coins needed to make change for V (or report impossible).

Subproblems. $OPT(v)$ = fewest coins needed to make change for amount v.

Optimal value. $OPT(V)$.

Multiway choice. To compute $OPT(v)$,

- Select a coin of denomination $d_i \leq v$ for some i.
- Use fewest coins to make change for $v - d_i$.

take best

optimal substructure

Dynamic programming recurrence.

$$OPT(v) = \begin{cases} \infty & \text{if } v < 0 \\ 0 & \text{if } v = 0 \\ \min\limits_{1 \leq i \leq n} \{ 1 + OPT(v - d_i) \} & \text{if } v > 0 \end{cases}$$

slide by Kevin Wayne

Interview Problem

💪 optional

Bottom-up DP implementation.

```
create an array opt[] of size V+1

for (v = 1 to V)
    opt[v] = INFINITY
    for (i = 1 to n)
        if (d[i] > v) continue
        else opt[v] = min(opt[v], 1 + opt[v - d[i]])
```

Running time. The bottom-up DP algorithm takes $\Theta(n\ V)$ time.

Note. Not polynomial in input size (and no poly-time algorithm is known).

n, $\log V$

adapted from a slide by Kevin Wayne

Activity Selection

Problem. Given n activities that require an exclusive use of a common resource, find the *maximum* number of non-overlapping activities.

Examples. Maximum number of jobs that can be done by a person.
Maximum number of events that can be held in a room.

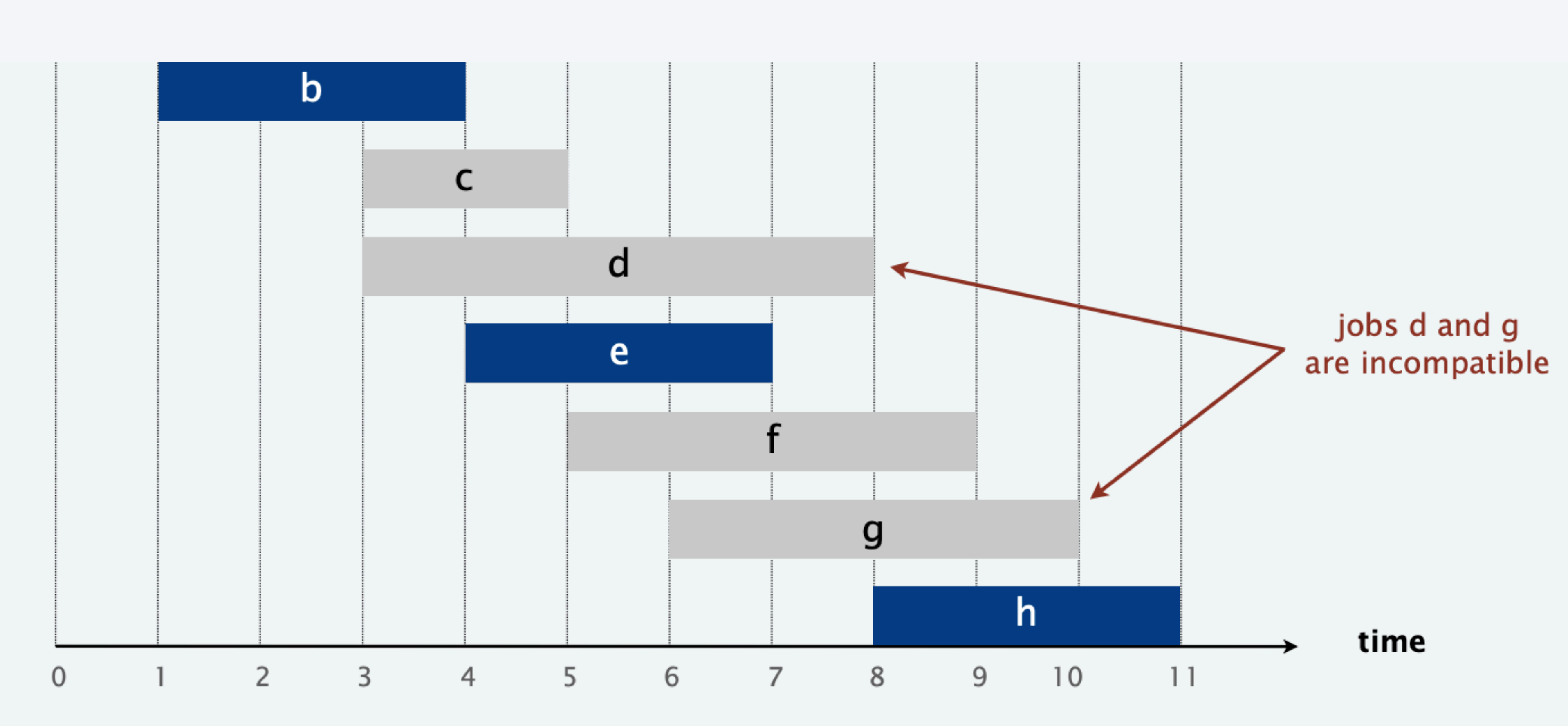

image by Kevin Wayne

Activity Selection

Problem. Given n activities that require an exclusive use of a common resource, find the *maximum* number of non-overlapping activities.

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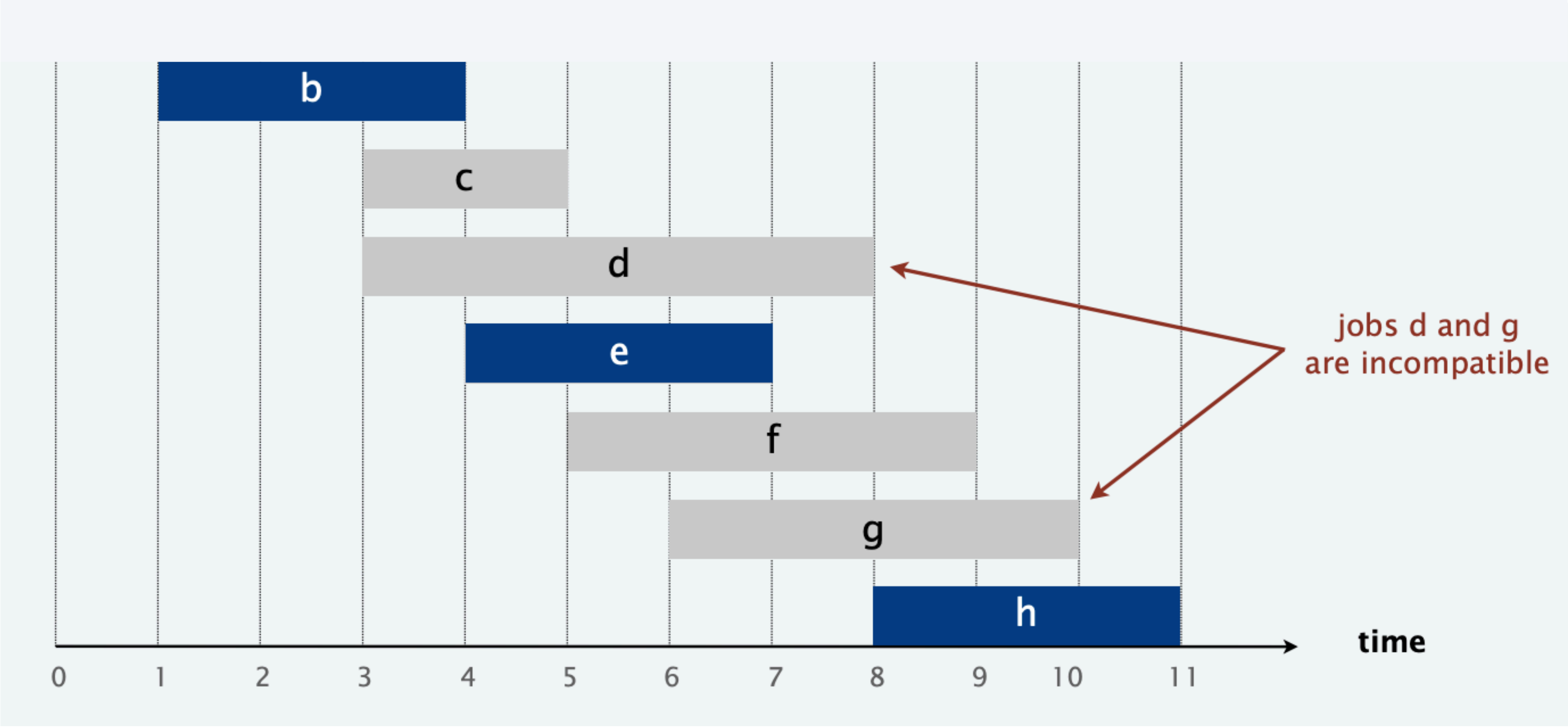

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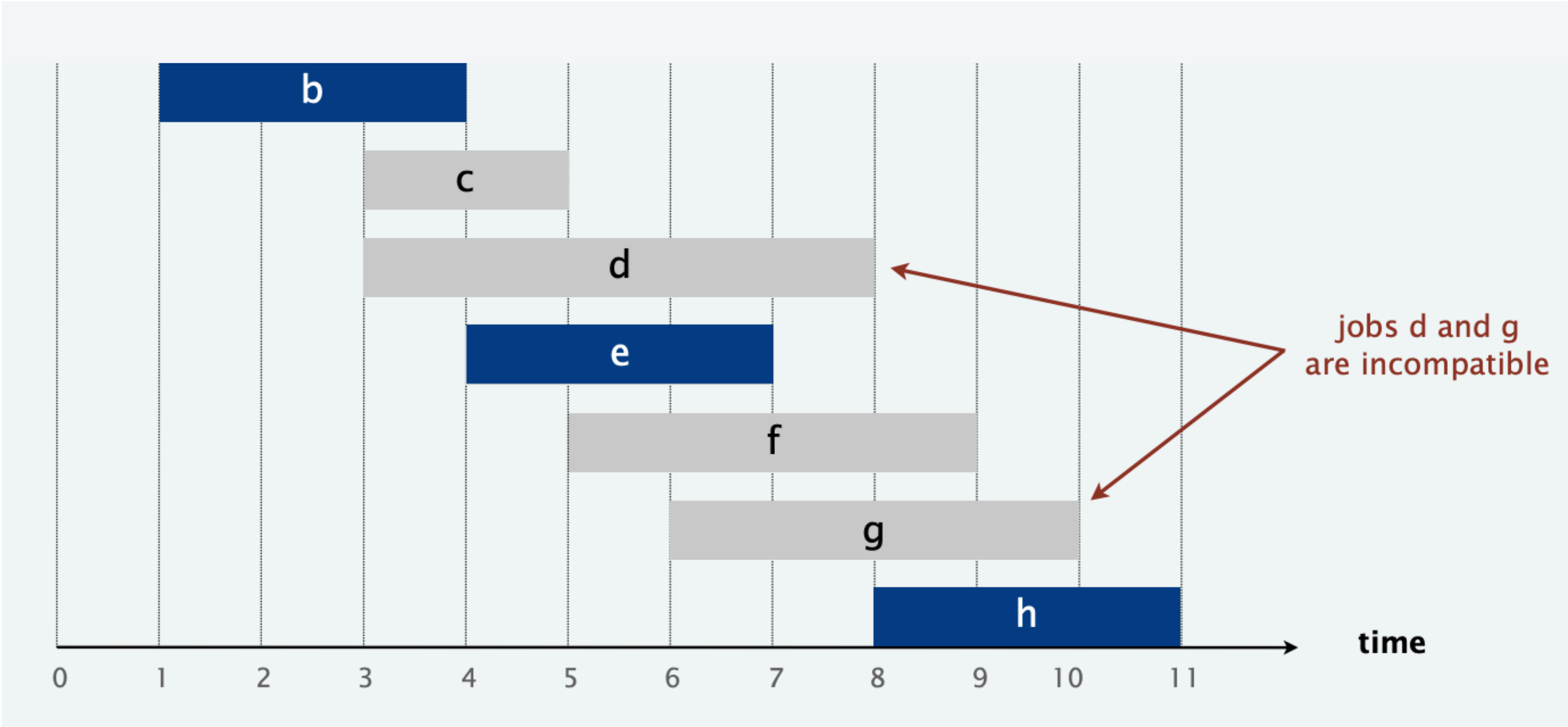

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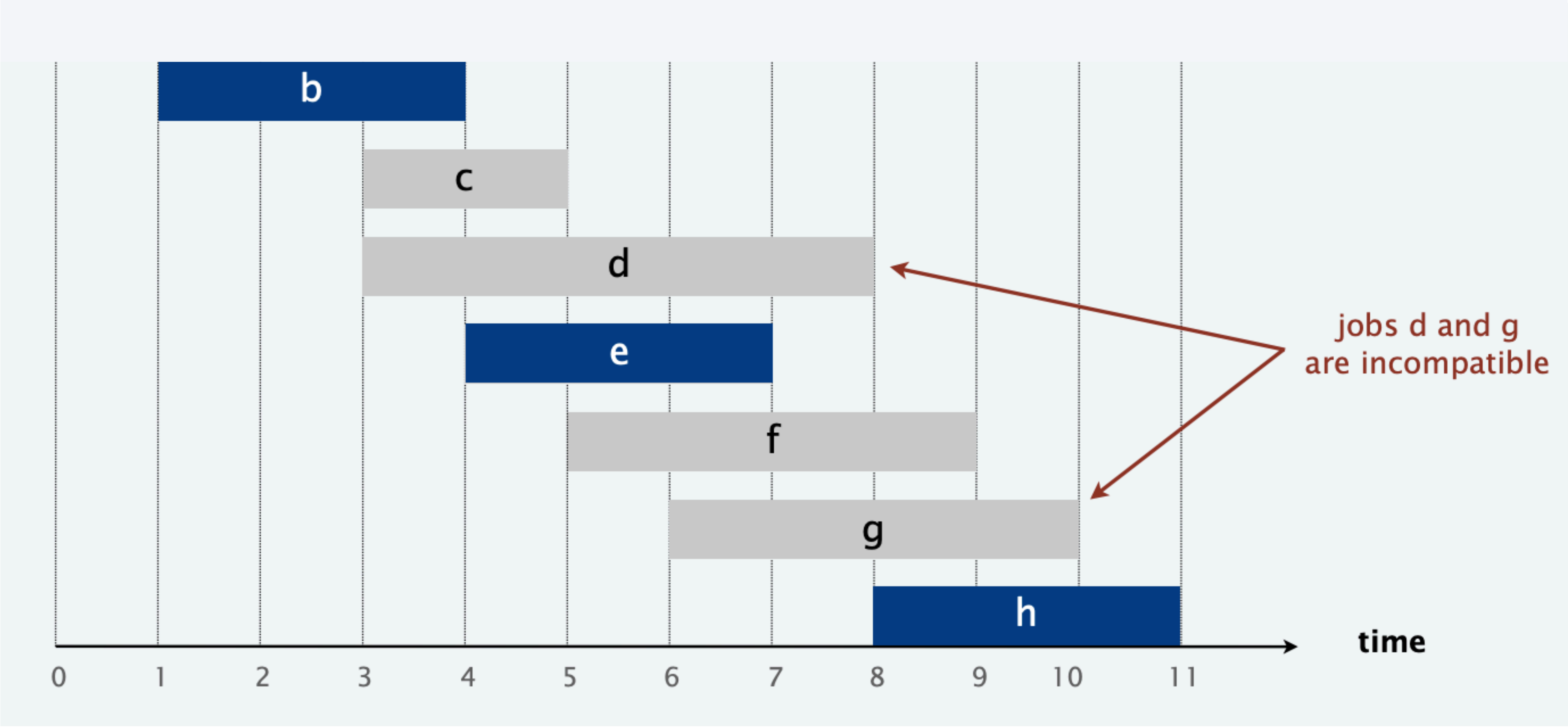

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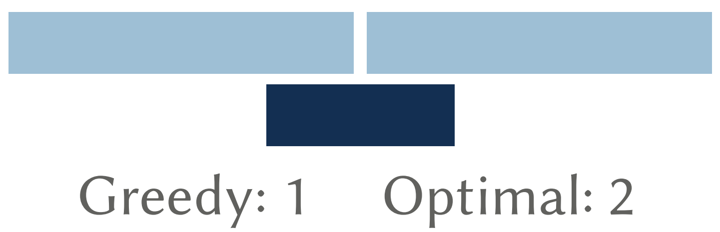

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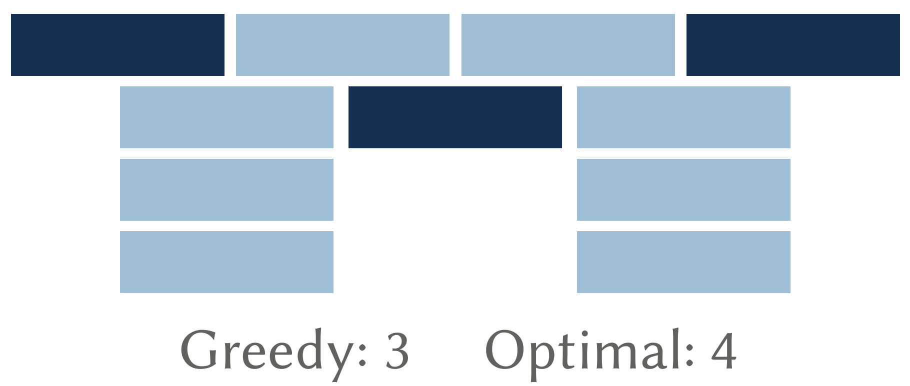

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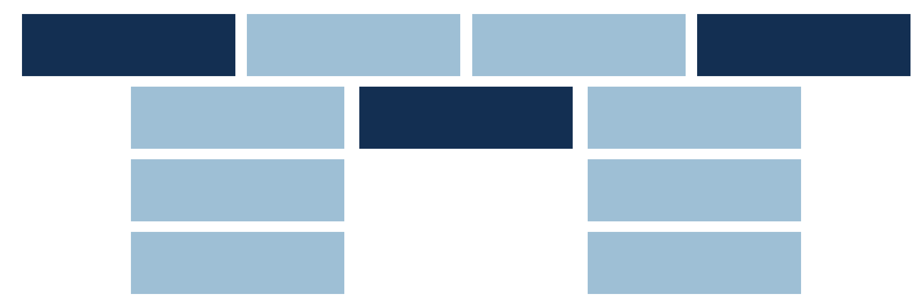

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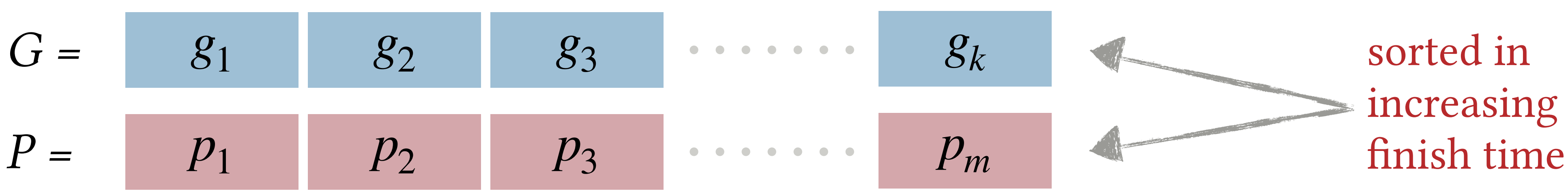

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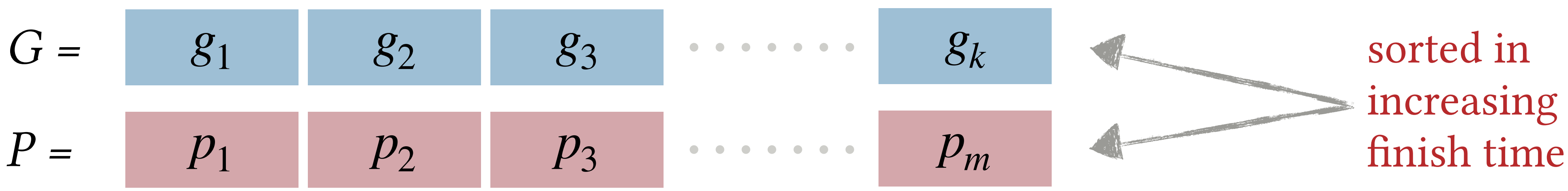

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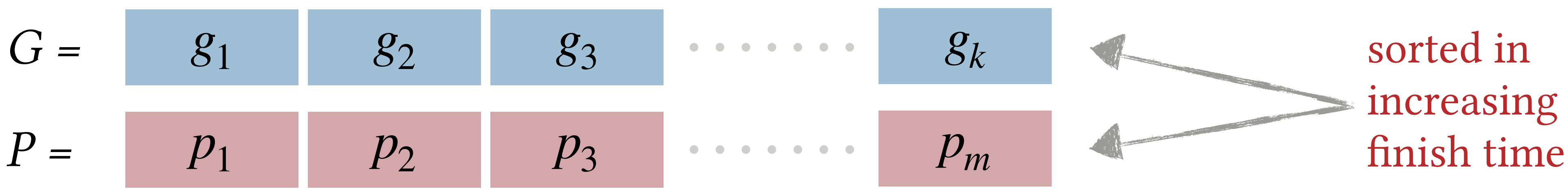

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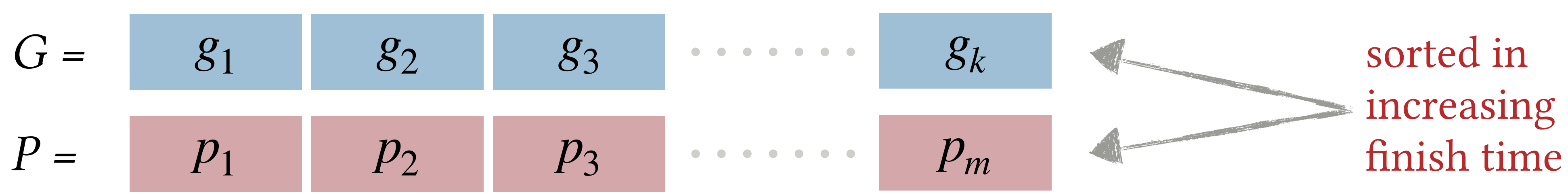

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- If $k = m$, then G is optimal.

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- If $k = m$, then G is optimal.
- If $k > m$, then P is not optimal.
- If $k < m$, then there is an activity that starts after g_k that G can still pick!

Greedy Proofs (General Pattern)

This pattern in proving the optimality of greedy solutions is commonly used:

- Assume that there is an optimal solution P and a greedy solution G.
- Show that we can always *exchange* the first choice in P with the first choice in G without making the solution worse.
- Show that the problem has an *optimal substructure* and thus the same argument applies to the solution of the subproblem after making the first choice.
- Show that P can't be better than G.

Activity Selection (Algorithm)

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SELECT(s[], f[], n)

Sort the activities by
increasing finish time

// Let k = the index of
the last taken activity
// Let A = the indices
of the taken activities
k = 0
Add k to A

for i=1 to n-1:
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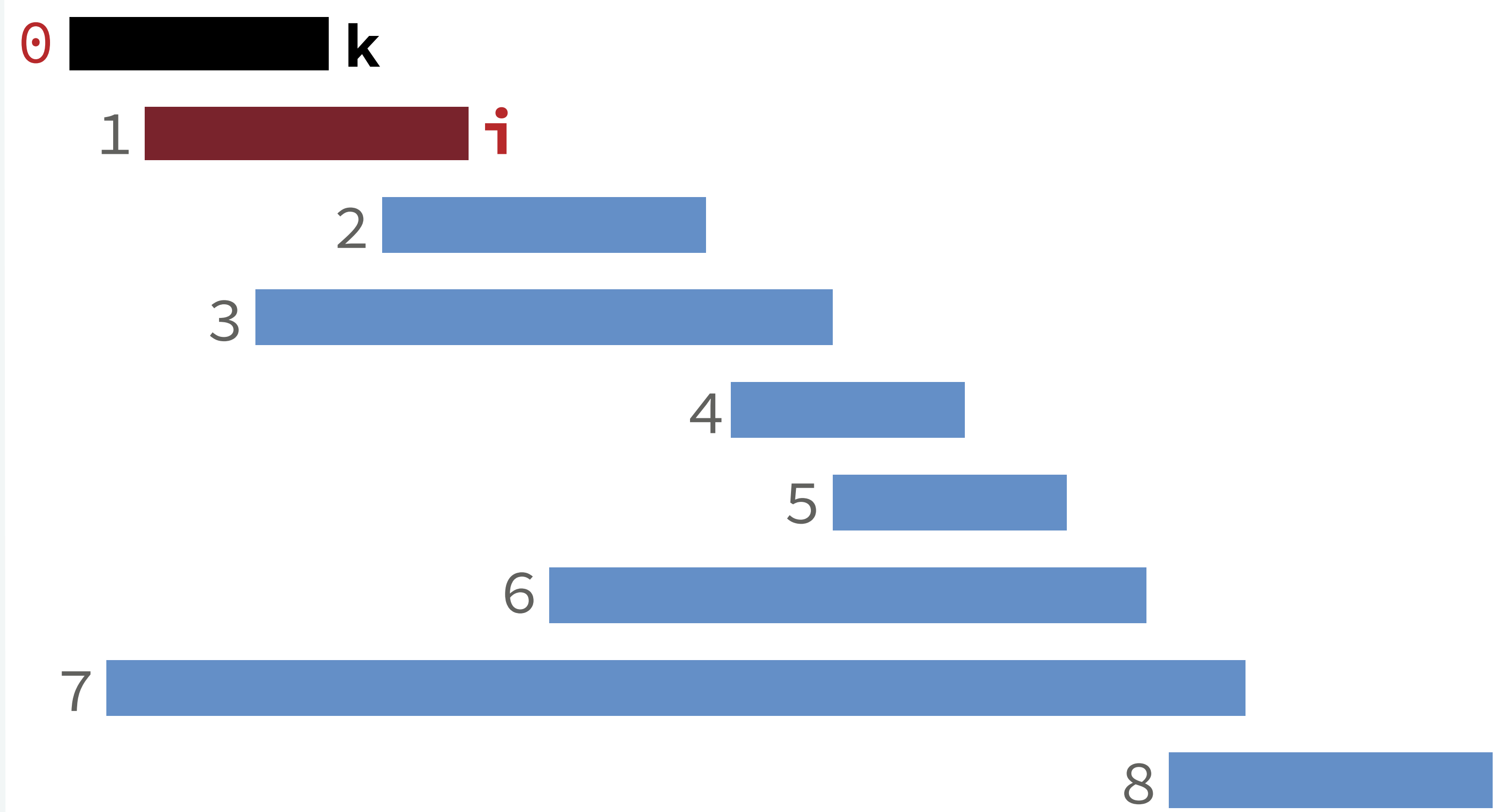

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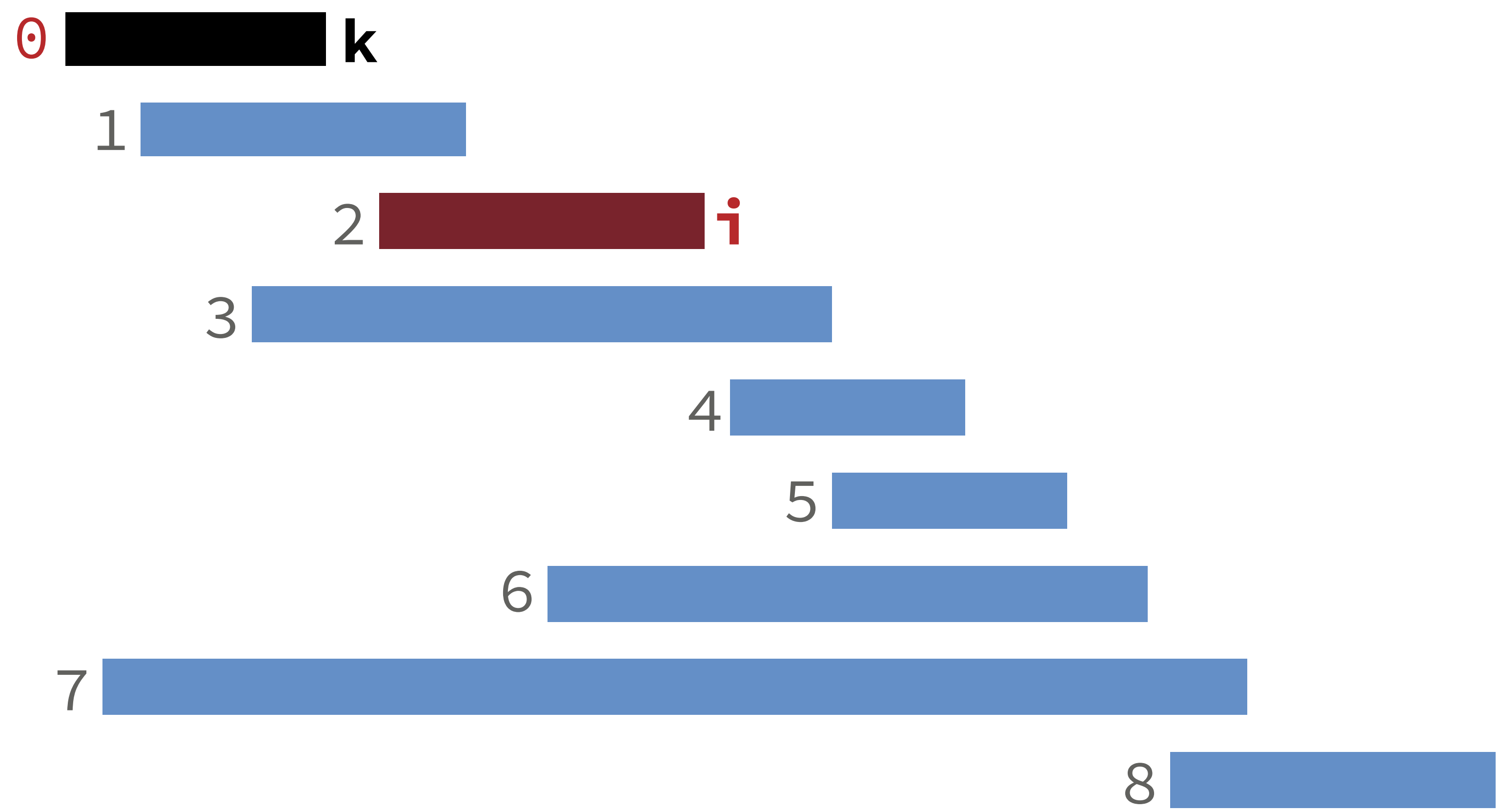

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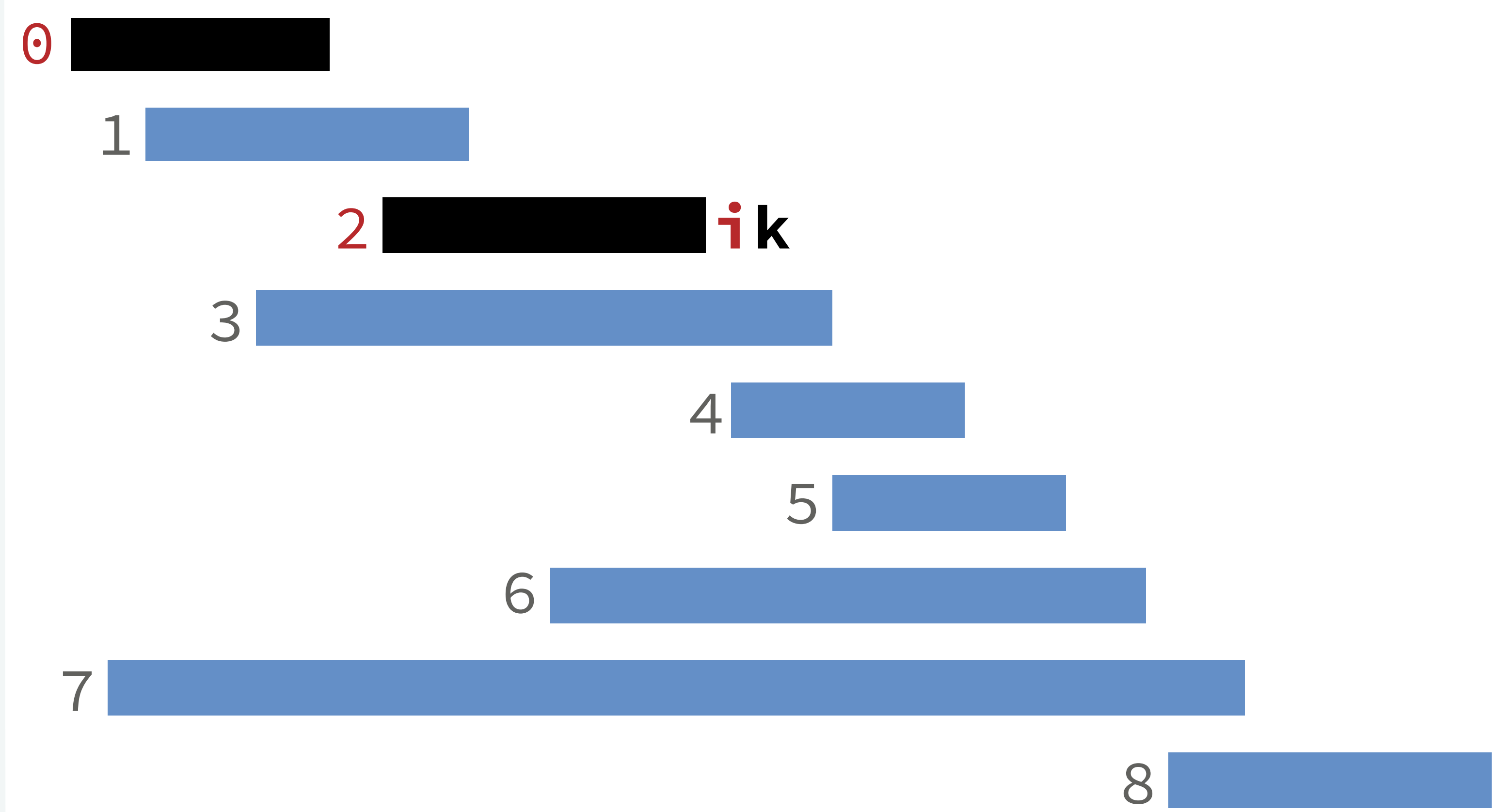

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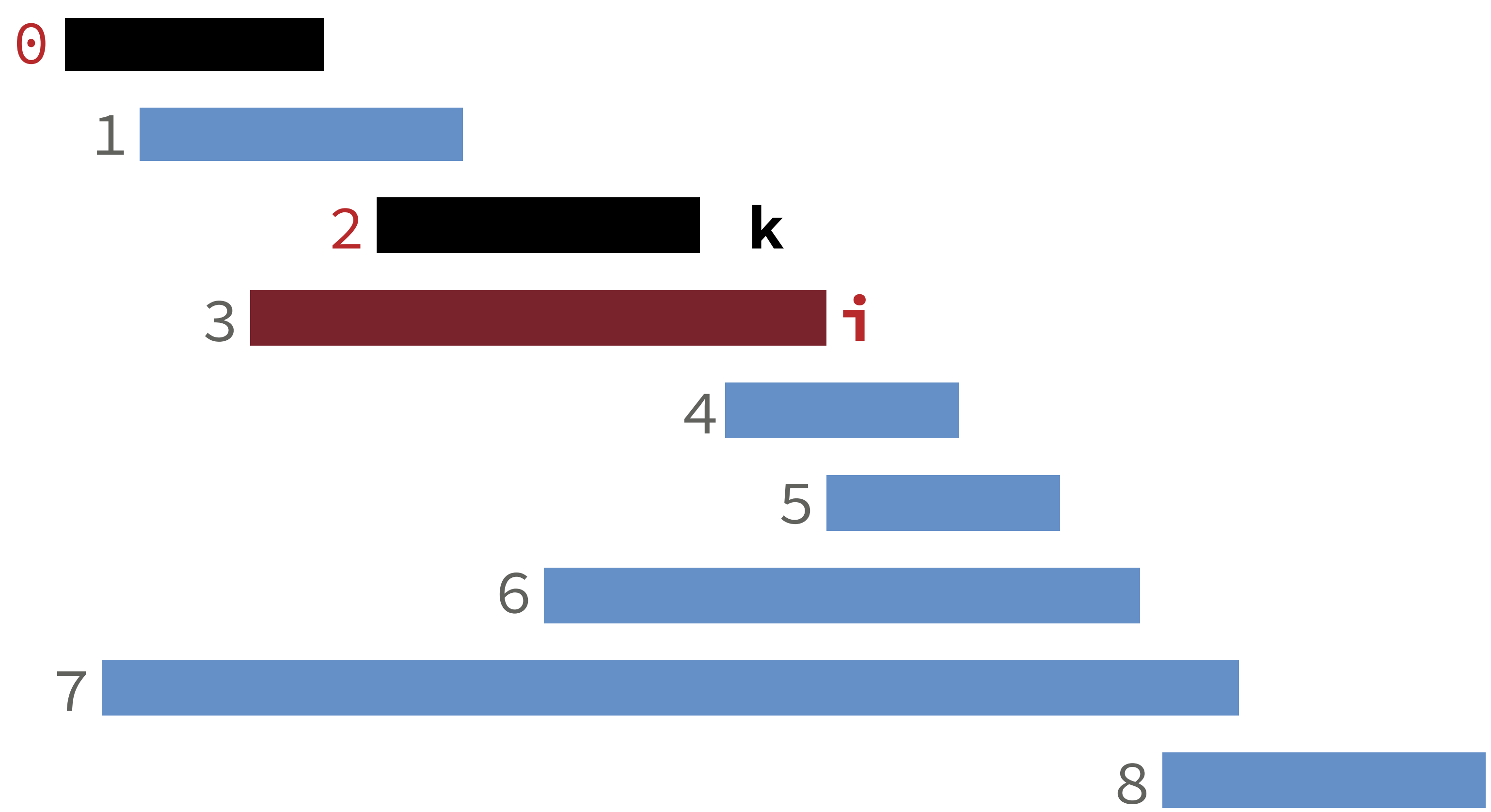

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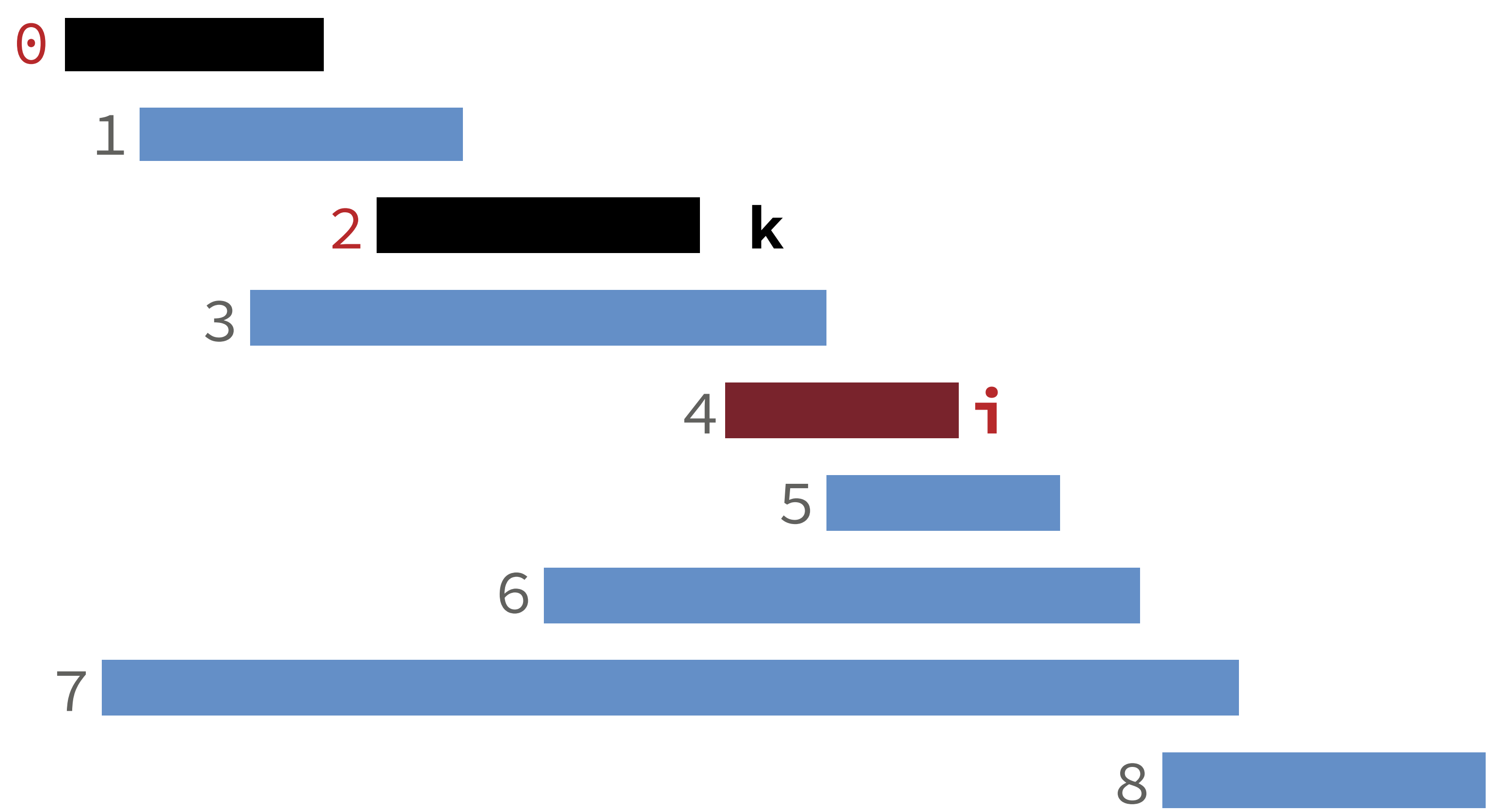

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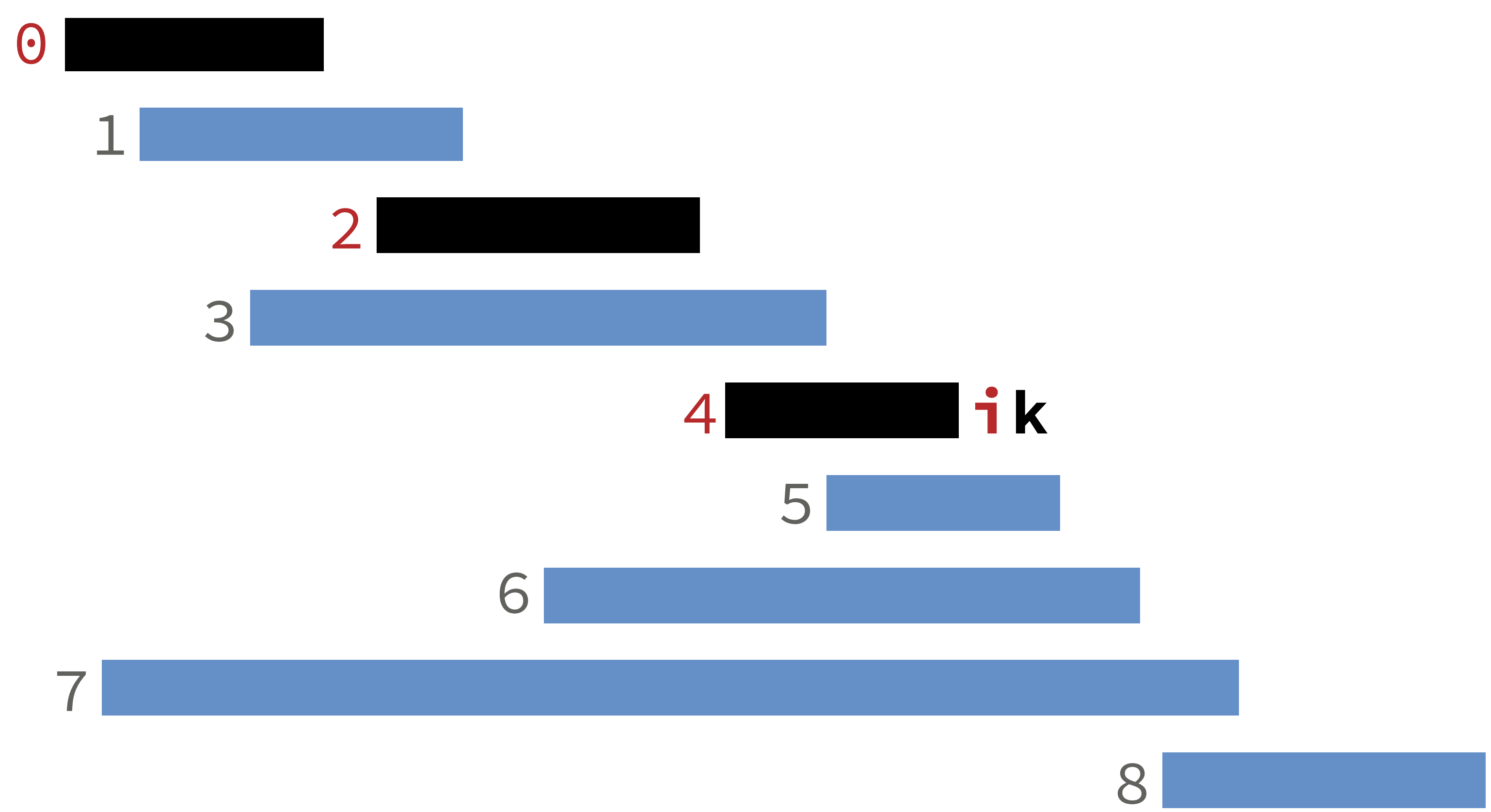

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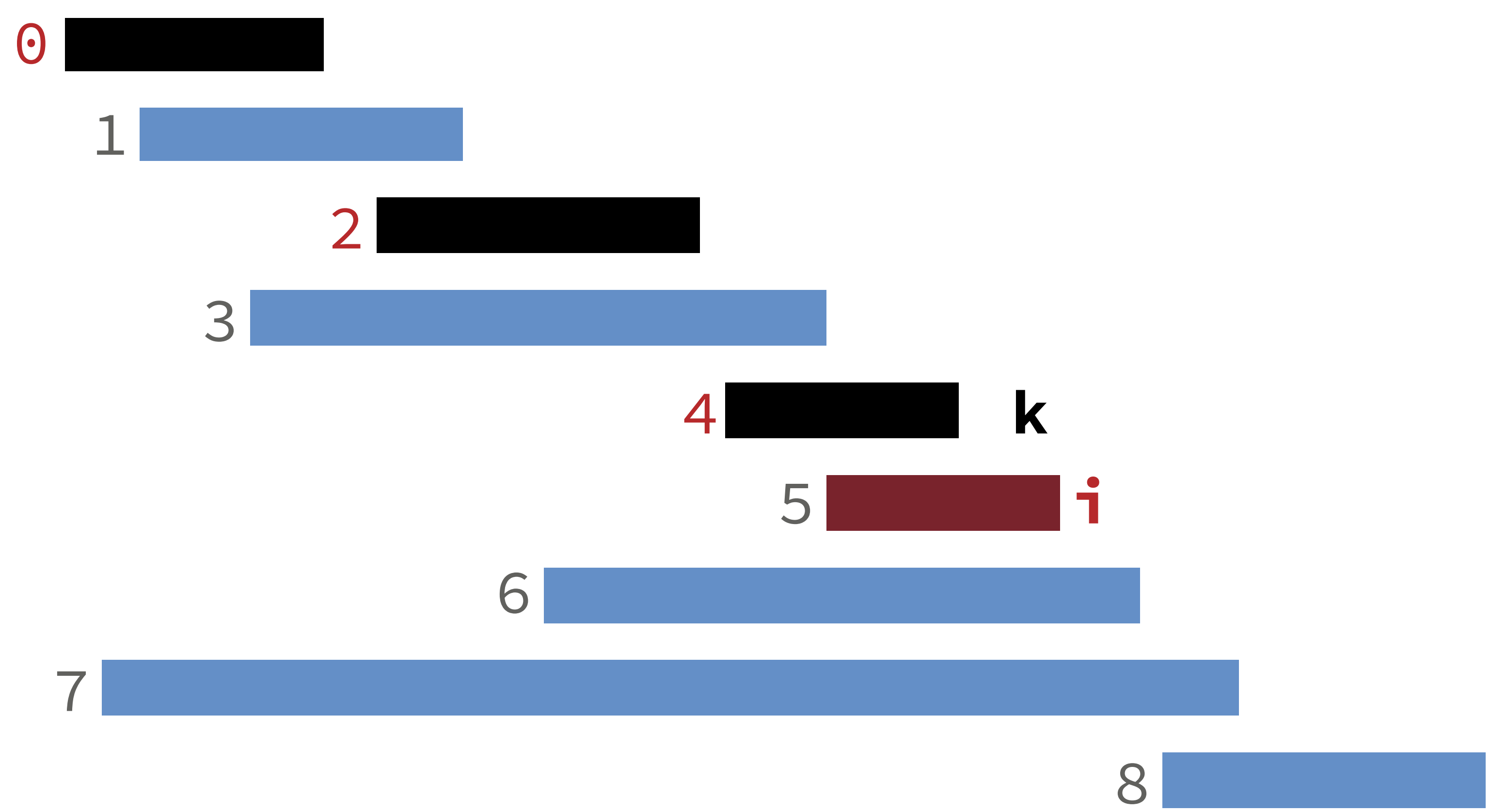

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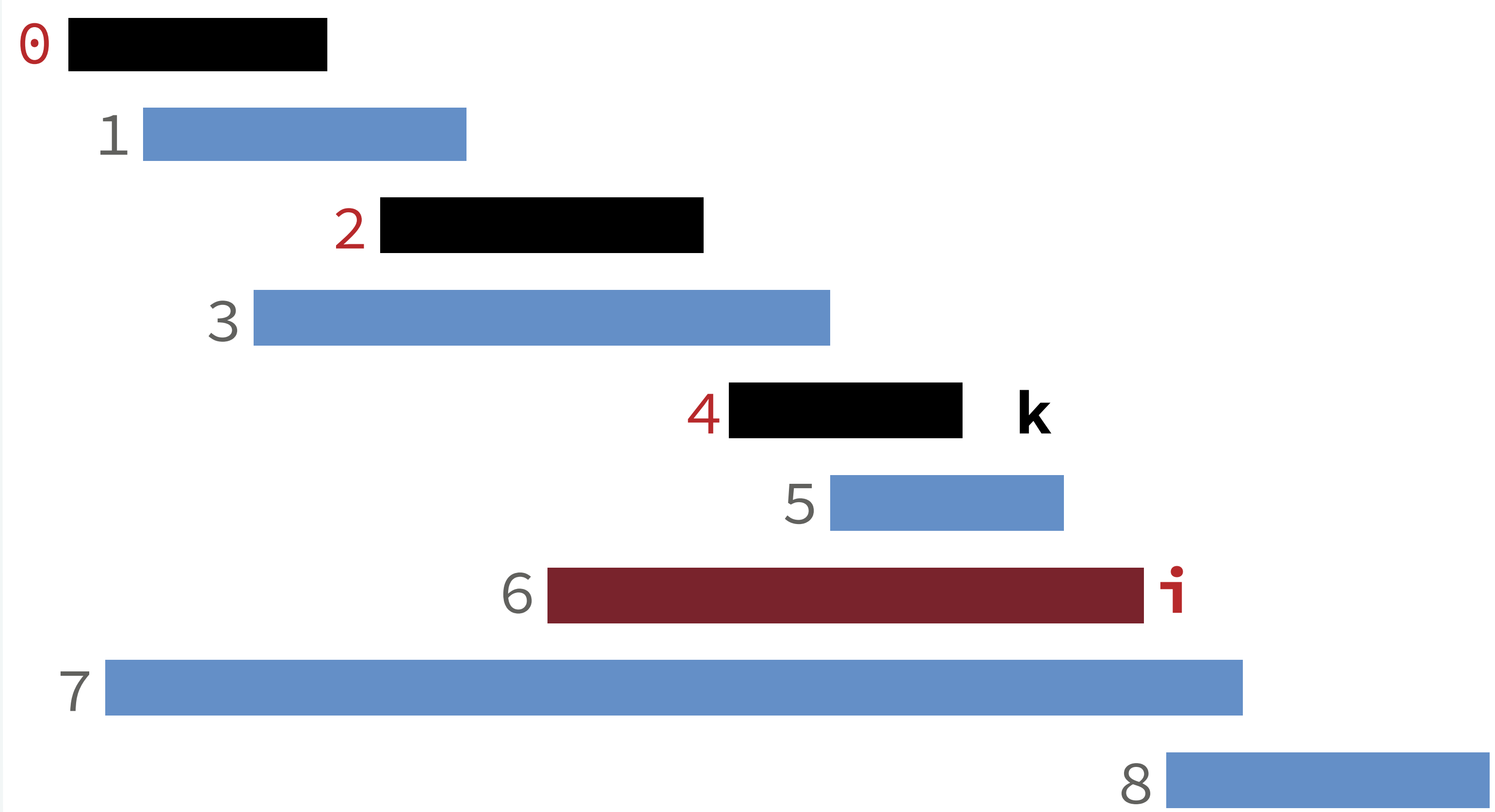

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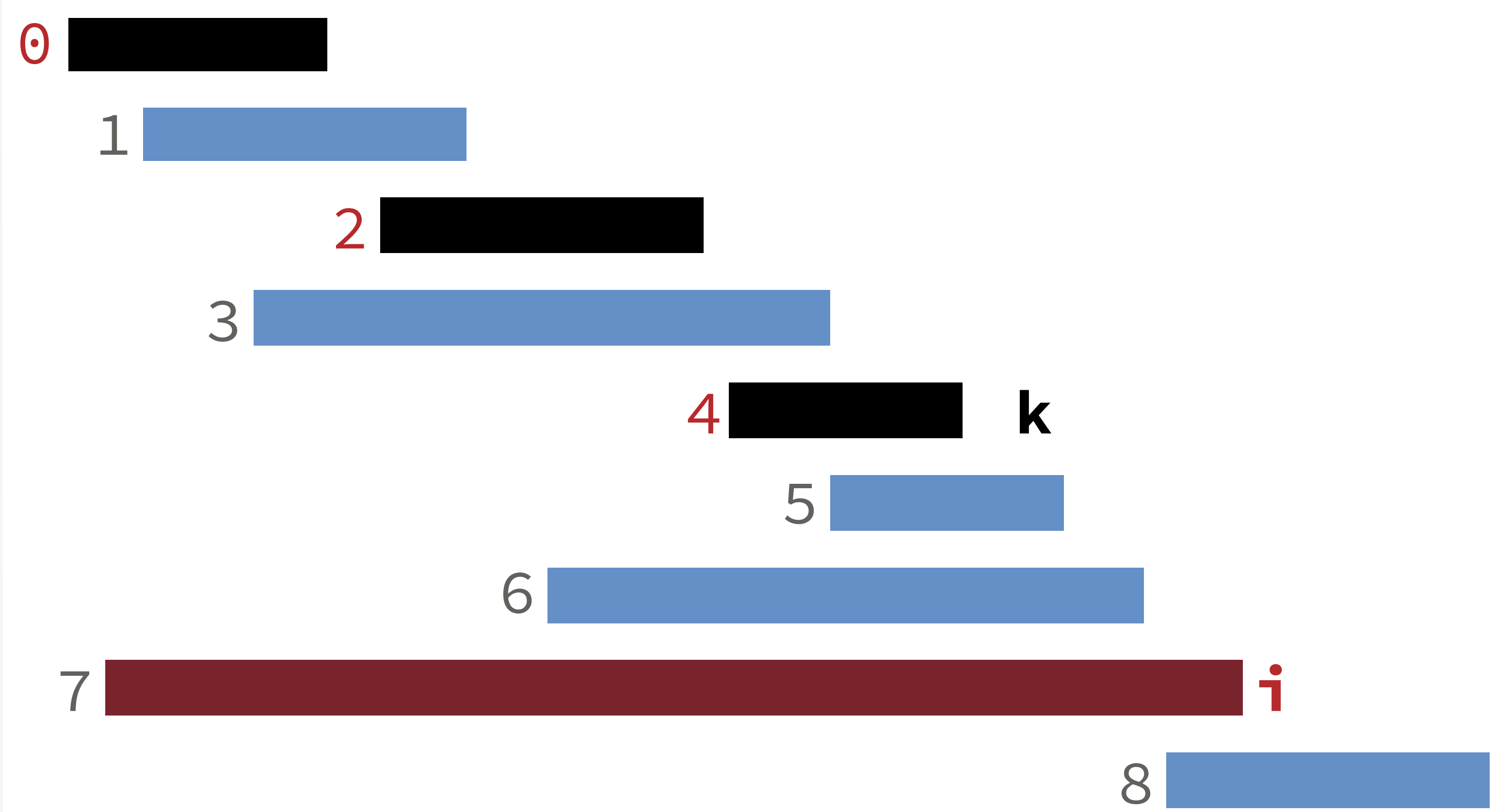

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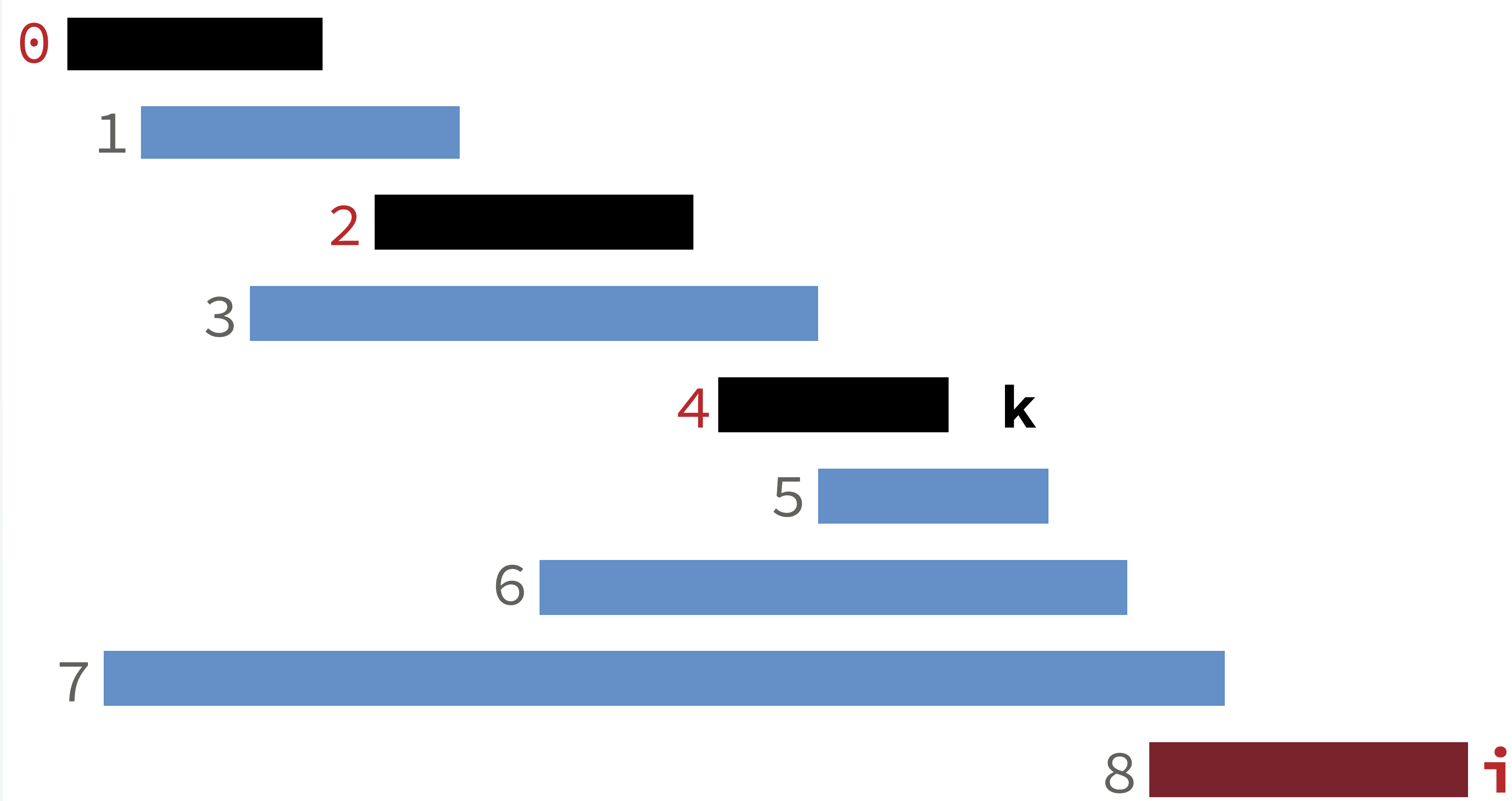

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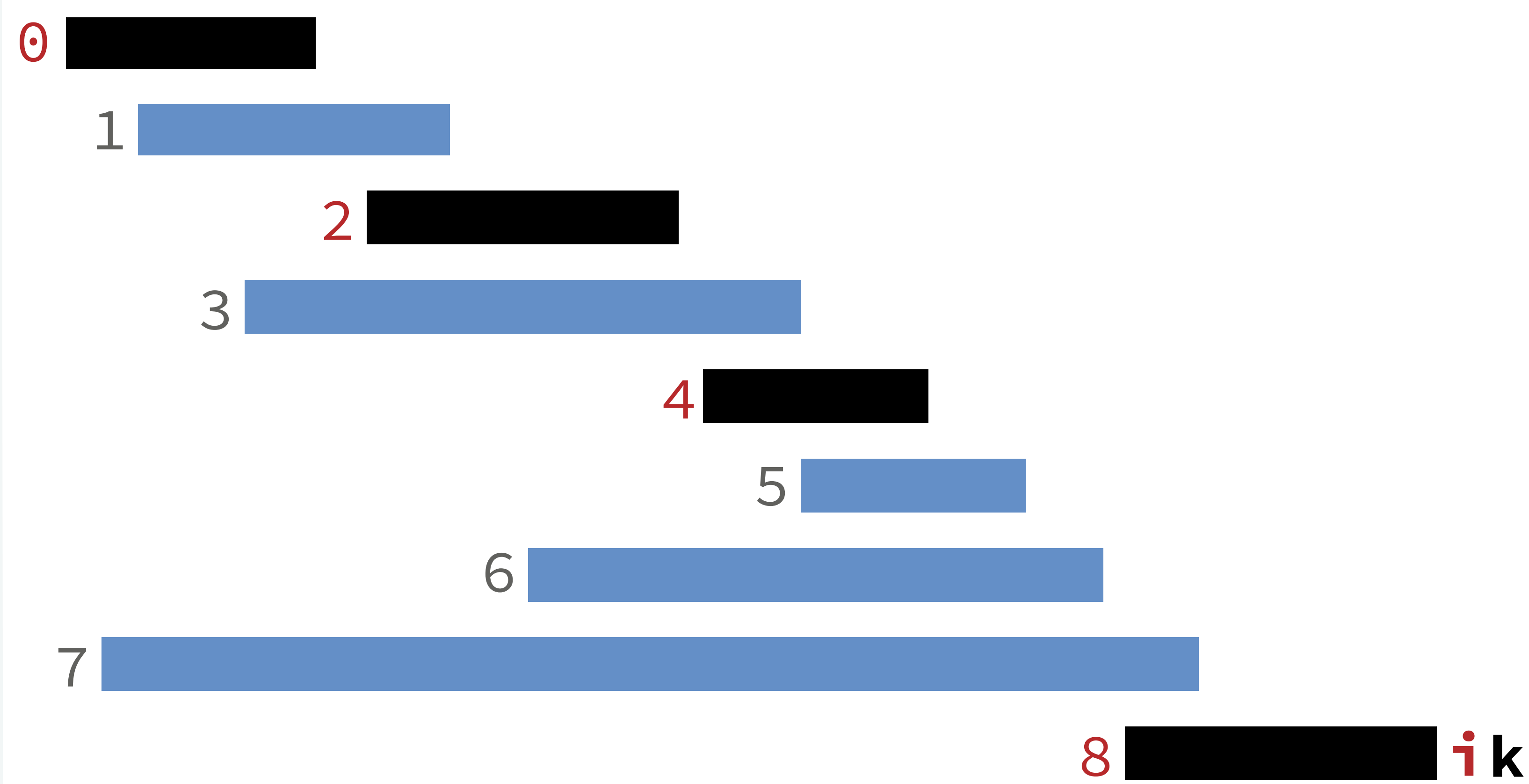

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Sort step → $\Theta(n \log n)$

Loop → $\Theta(n)$

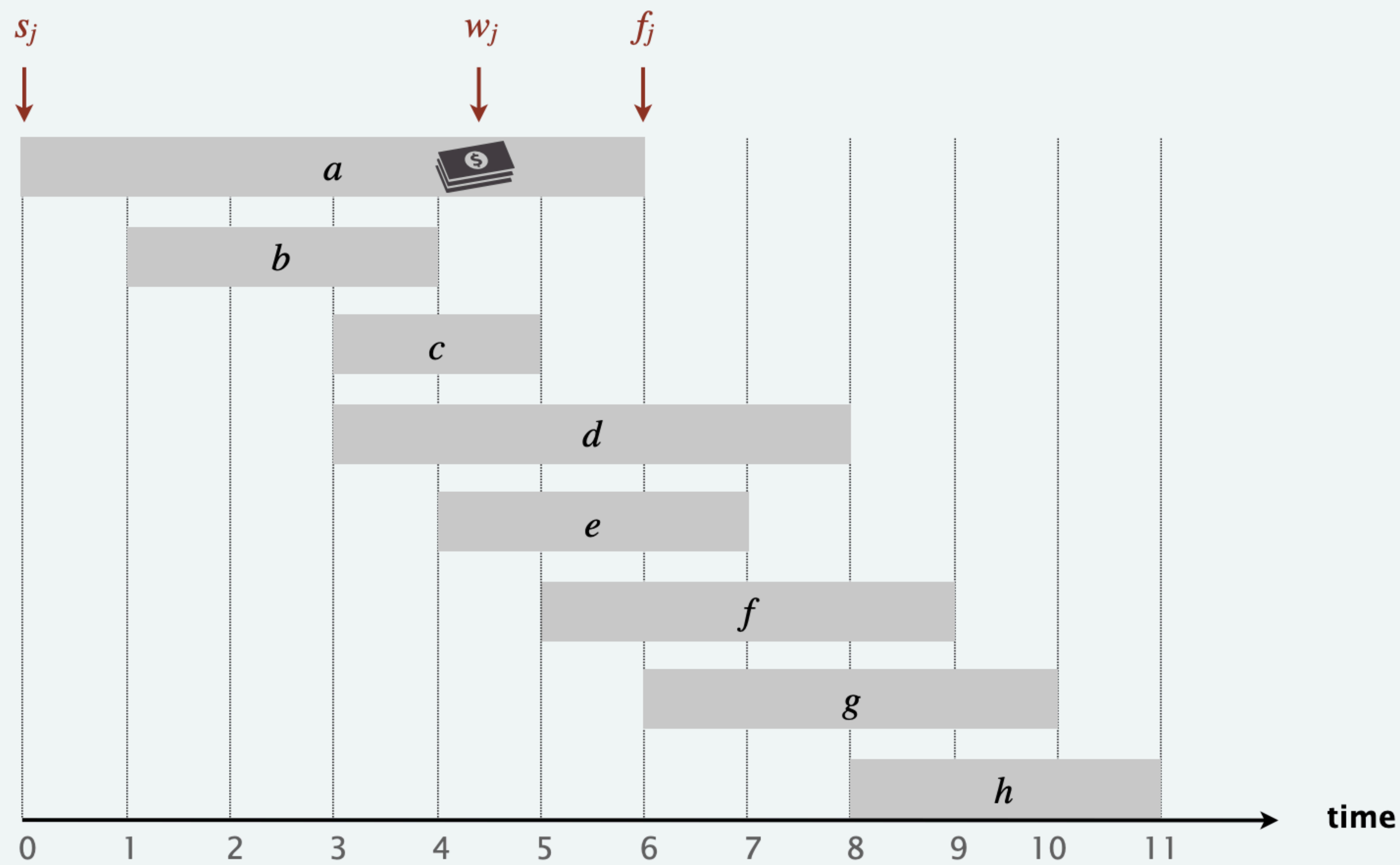
💪 optional

Interview Problem

💪 optional

Weighted interval scheduling

- Job j starts at s_j, finishes at f_j, and has weight $w_j > 0$.
- Two jobs are compatible if they don't overlap.
- Goal: find max-weight subset of mutually compatible jobs.



slide by Kevin Wayne

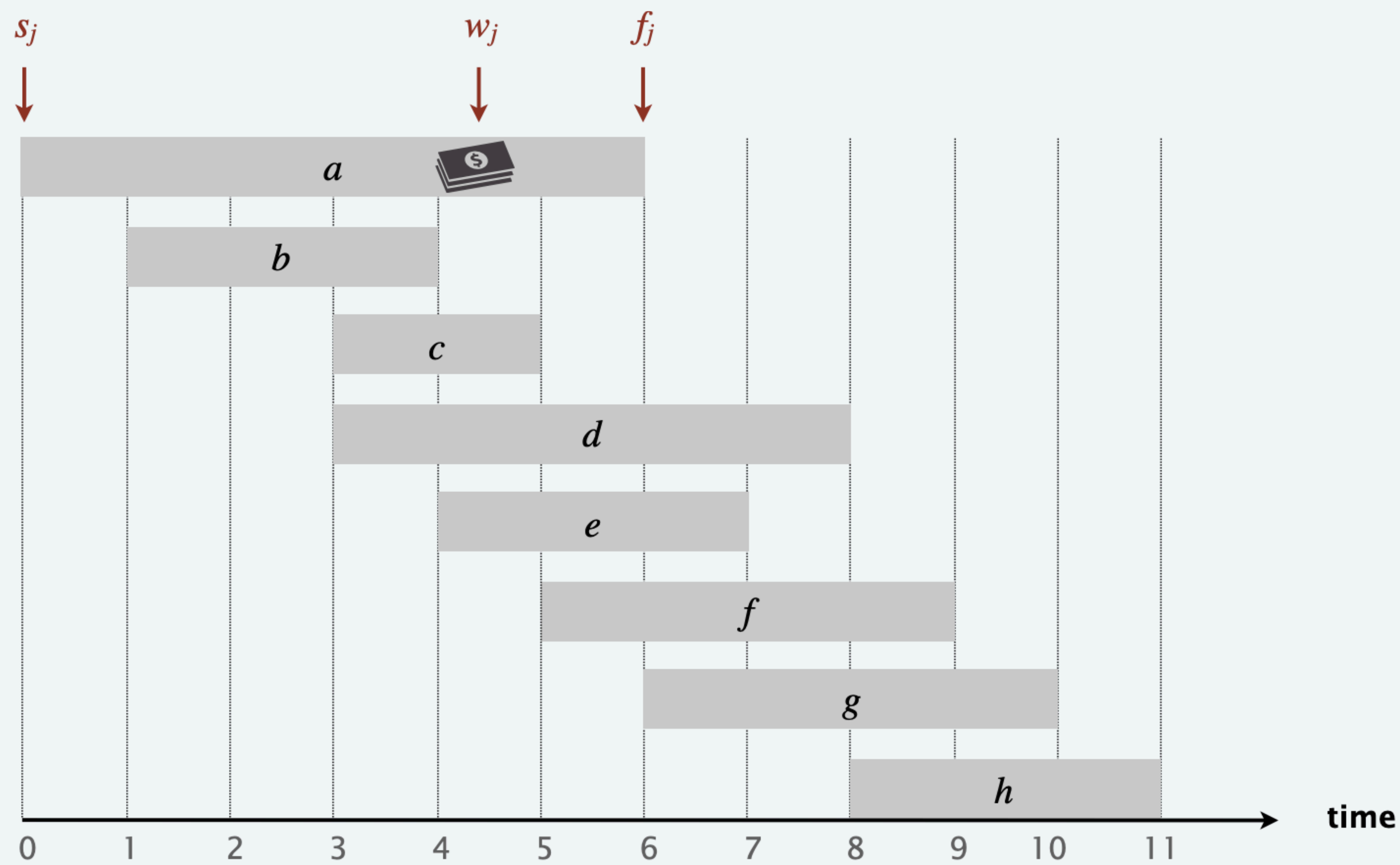
Weighted interval scheduling

Convention. Jobs are in ascending order of finish time: $f_1 \le f_2 \le \ldots \le f_n$.

Def. $p(j)$ = largest index $i < j$ such that job i is compatible with j.

Ex. $p(8) = 1, p(7) = 3, p(2) = 0$.

i is leftmost interval that ends before j begins



Interview Problem

💪 optional

Dynamic programming: binary choice

Def. $OPT(j)$ = max weight of any subset of mutually compatible jobs for subproblem consisting only of jobs $1, 2, \ldots, j$.

Goal. $OPT(n)$ = max weight of any subset of mutually compatible jobs.

Case 1. $OPT(j)$ does not select job j.

- Must be an optimal solution to problem consisting of remaining jobs $1, 2, \ldots, j-1$.

Case 2. $OPT(j)$ selects job j.

- Collect profit w_j.
- Can't use incompatible jobs $\{ p(j)+1, p(j)+2, \ldots, j-1 \}$.
- Must include optimal solution to problem consisting of remaining compatible jobs $1, 2, \ldots, p(j)$.

Bellman equation.
$$OPT(j) = \begin{cases} 0 & \text{if } j = 0 \\ \max\{ OPT(j-1),\ w_j + OPT(p(j)) \} & \text{if } j > 0 \end{cases}$$

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