CS11212 - **Spring** 2022

Data Structures & Introduction to **Algorithms**

Data Structures

Trees: Definitions and Properties

Ibrahim Albluwi

A Set ADT

Problem. Design a data structure to support the following operations:

- `insert(val)   // add val to the set if it is not already in the set.`
- `remove(val)   // remove val from the set of items.`
- `contains(val) // check if val belongs to the set.`

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Candidate implementations.

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Unordered DLL Unordered SLL			

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both need searching in the list

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Unordered DLL	`O(n)`	`O(n)`	`O(n)`
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Maintaining order does not help!

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Ordered DLL	`O(n)`	`O(n)`	`O(n)`	Maintaining order does not help!
Ordered SLL	`O(n)`	`O(n)`	`O(n)`	
Unordered Array	`O(n)`	`O(n)`	`O(n)`	Bad!
Ordered Array	`O(n)`	`O(n)`	`O(log n)`	Great! insert/remove are bad!

A Set ADT

What is going on?

Great!

- Linked lists support efficient insertion and deletion.
  Provided that there is a pointer to the insertion or deletion position.
- Arrays support efficient search.
  Provided that the array is sorted.

Bad!

- Linked lists do not support efficient search.
  Sorted linked lists do not support efficient binary search because it requires direct access.
- Arrays do not support efficient insertion and removal at any position.
  Elements need to be shifted.

A Set ADT

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Can we achieve the good of the two?

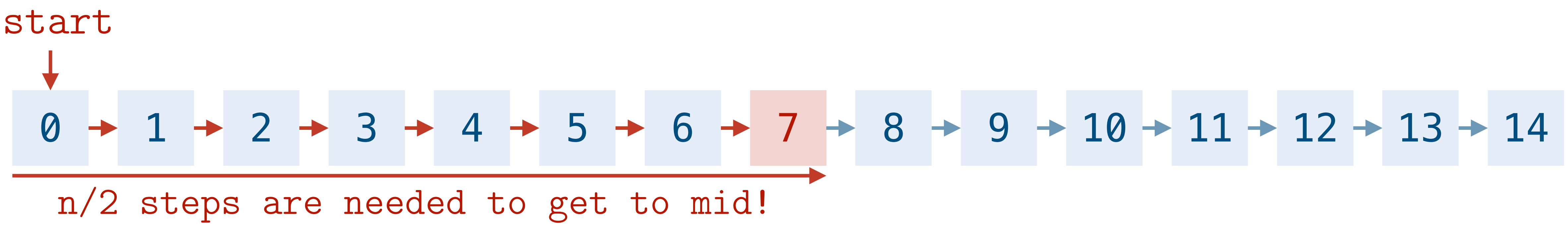
Efficient search (as in binary search)
Efficient insertion / removal (given a pointer to the insertion / removal position)

Binary Search on Linked Lists?

0 → 1 → 2 → 3 → 4 → 5 → 6 → 7 → 8 → 9 → 10 → 11 → 12 → 13 → 14

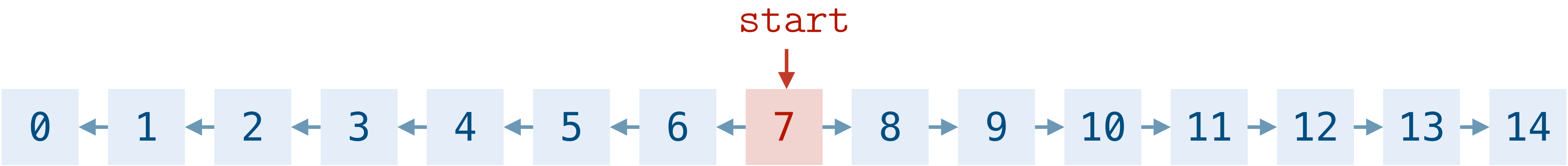
Can we perform *binary search* on a sorted linked list?

Binary Search on Linked Lists?



$O(n)$ operations to get to the middle element!

Binary Search on Linked Lists?



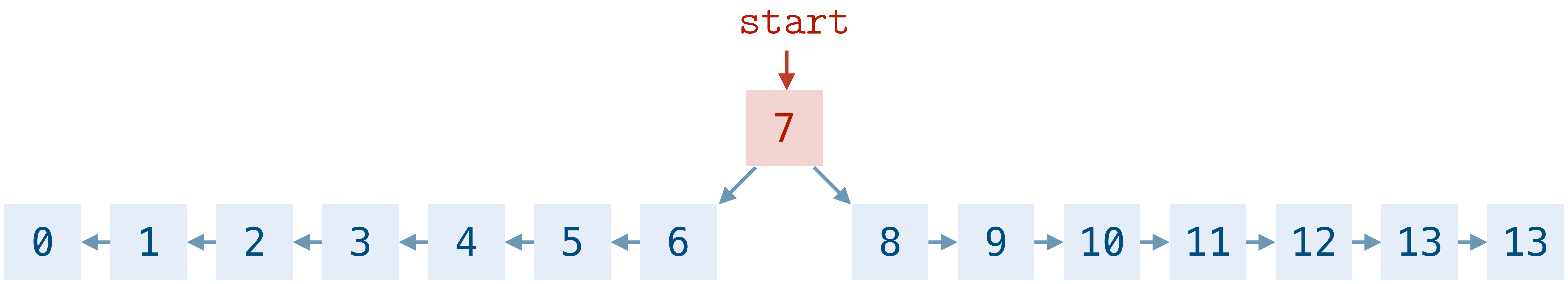
`start`

`0 ← 1 ← 2 ← 3 ← 4 ← 5 ← 6 ← 7 → 8 → 9 → 10 → 11 → 12 → 13 → 14`

Idea: Maintain a pointer to the middle element.

The middle element is now accessible in $O(1)$.

Binary Search on Linked Lists?



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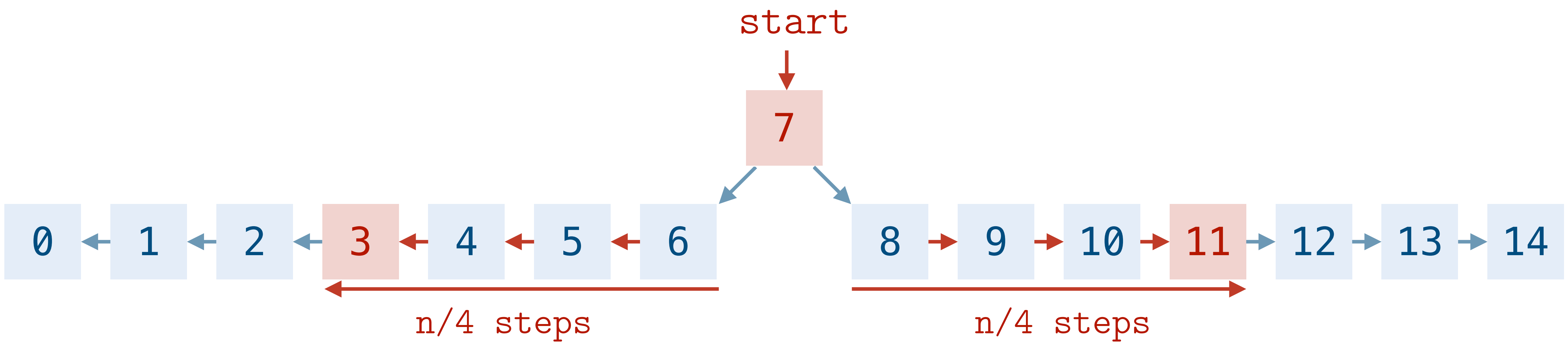
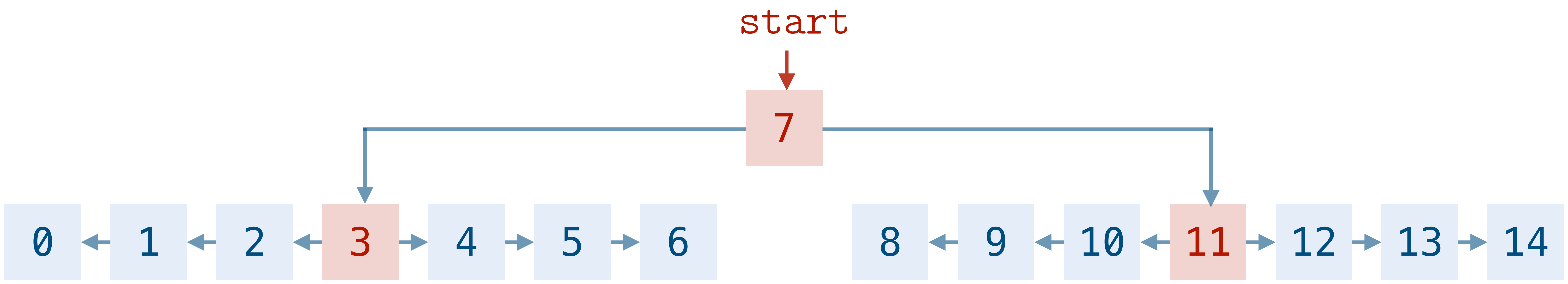


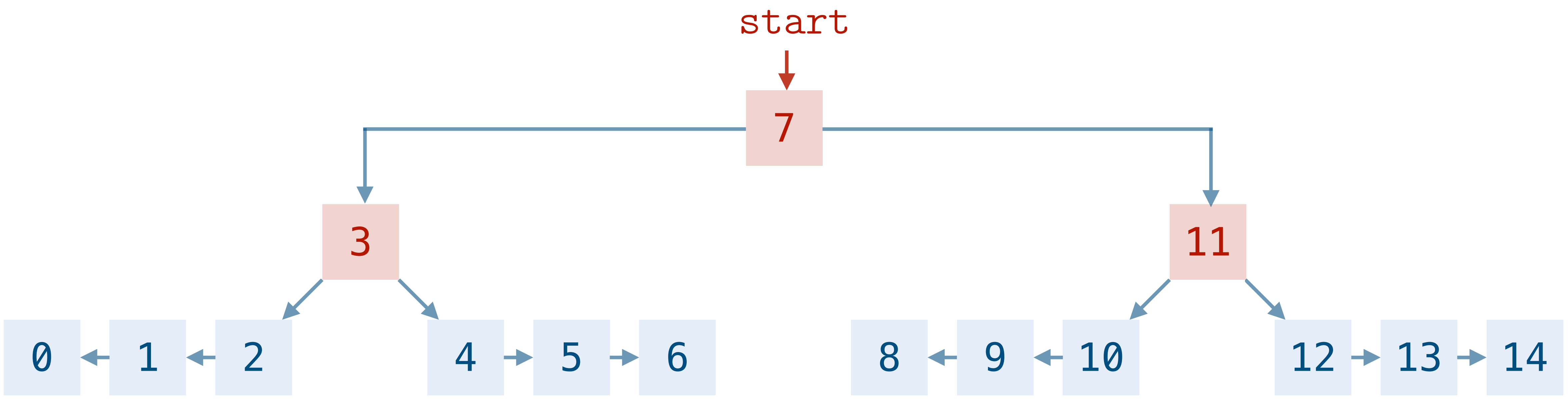
Still $O(n)$ to get to the middle of the left half or the middle of the right half!

Binary Search on Linked Lists?



Idea (*again*): Maintain a pointer to the middle elements!

Binary Search on Linked Lists?



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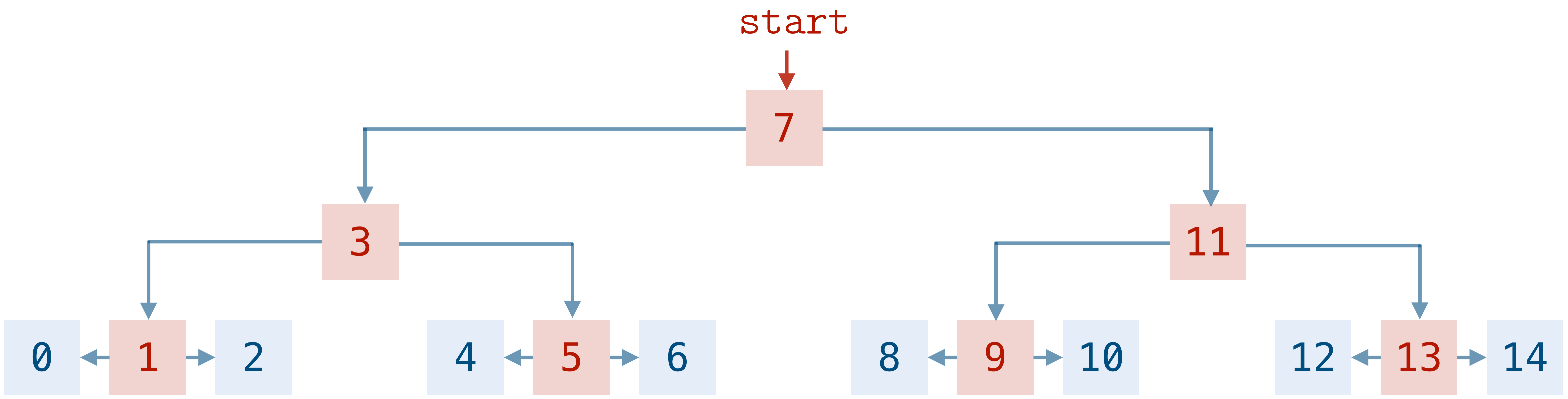

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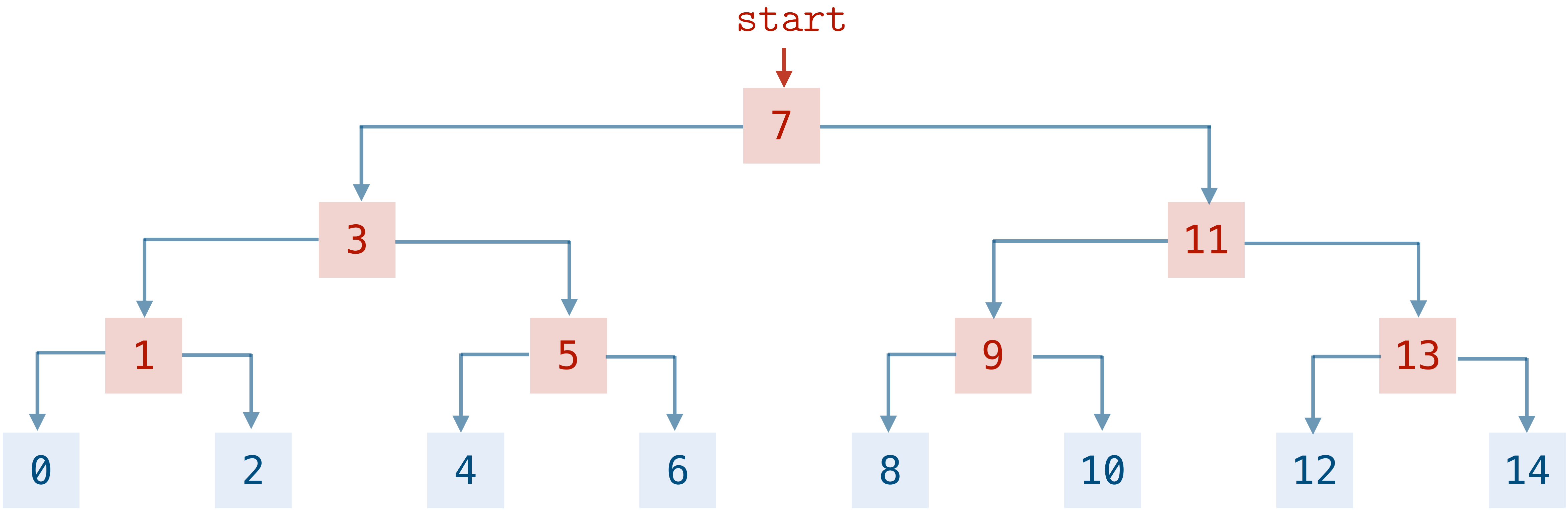

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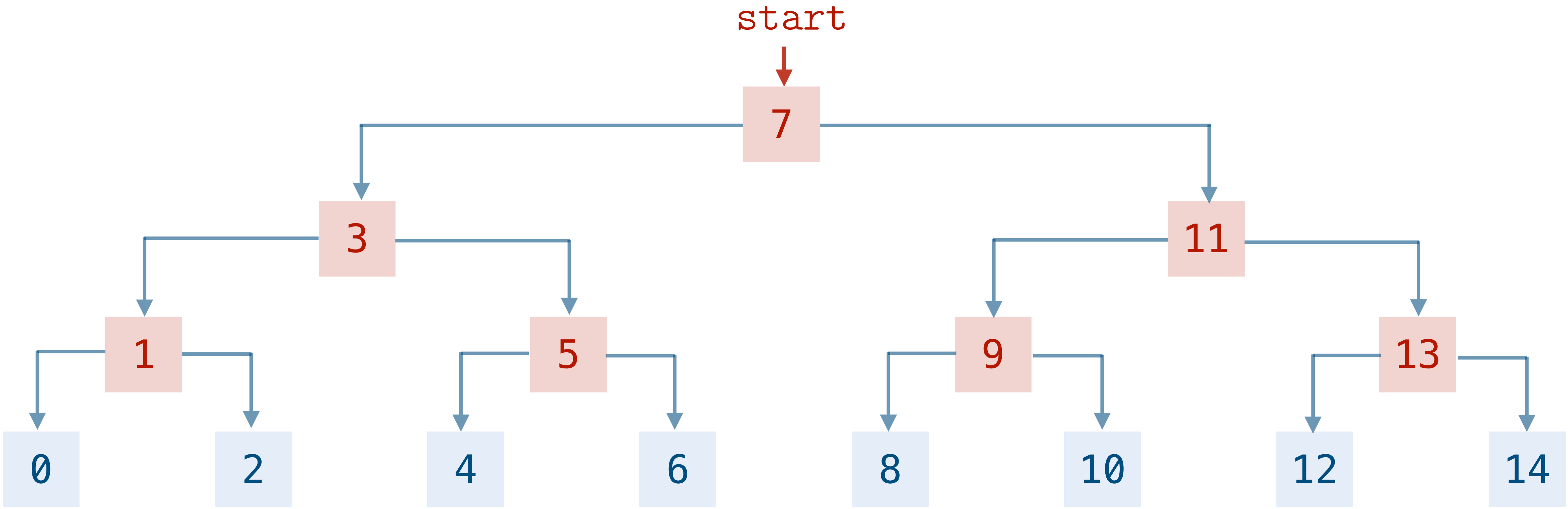
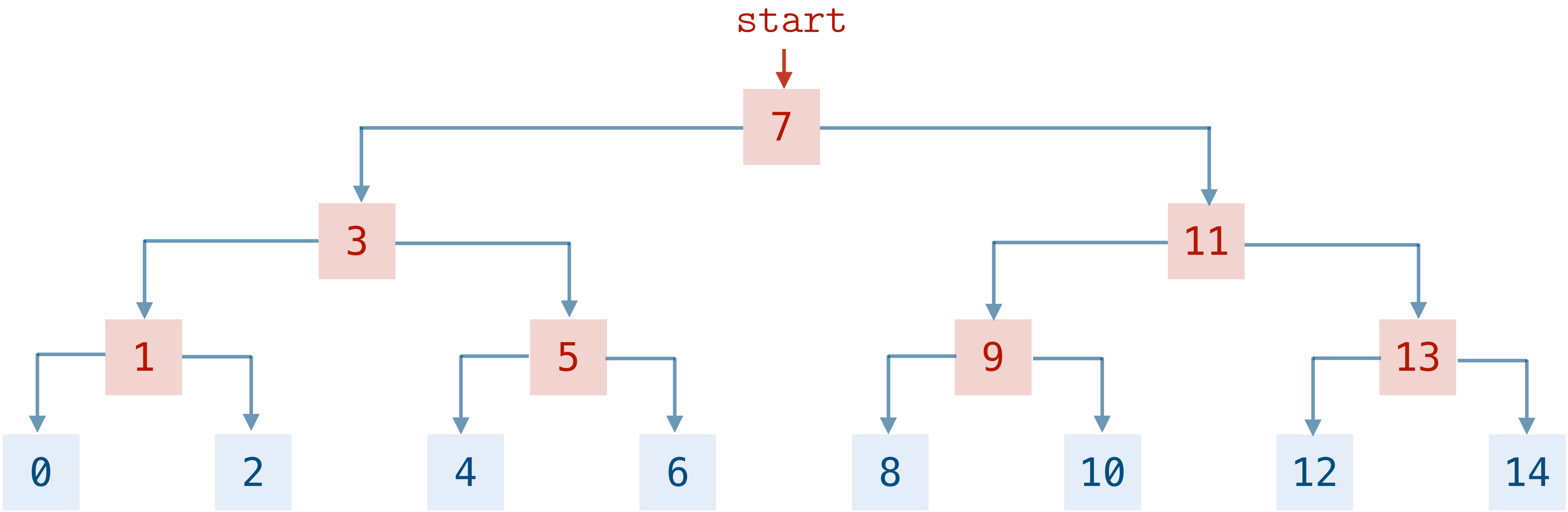


A Binary Search Tree!

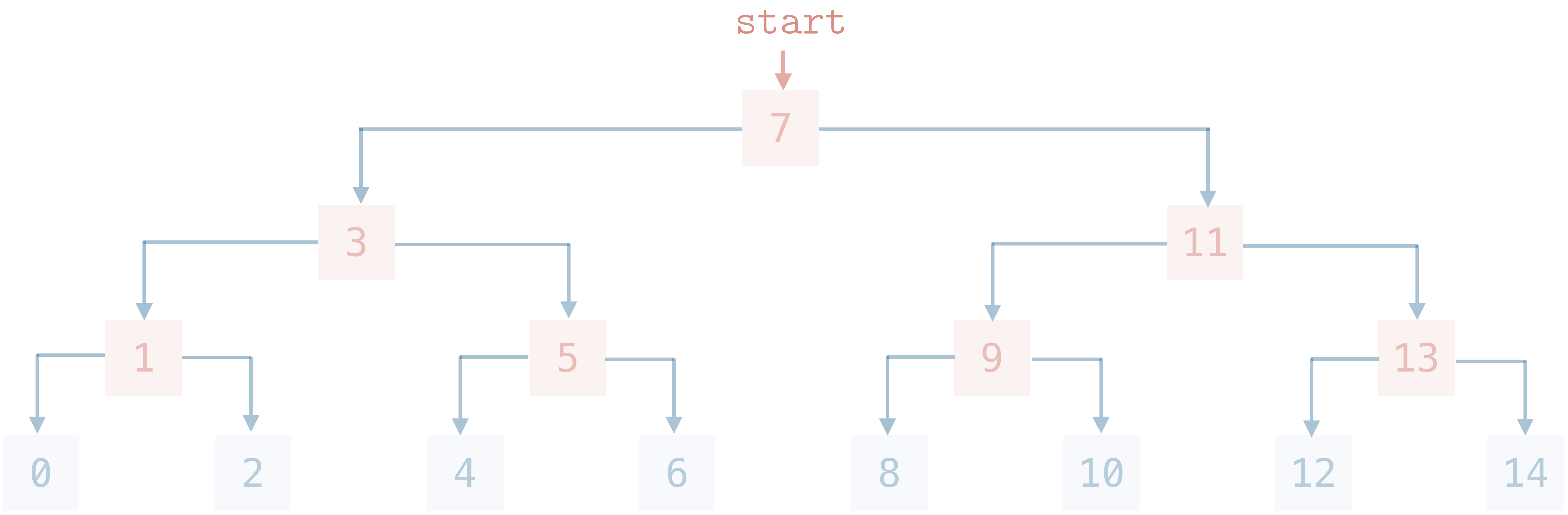
Proposed Implementation



A Binary Search Tree!

We need to build a linked *tree structure.*

Proposed Implementation



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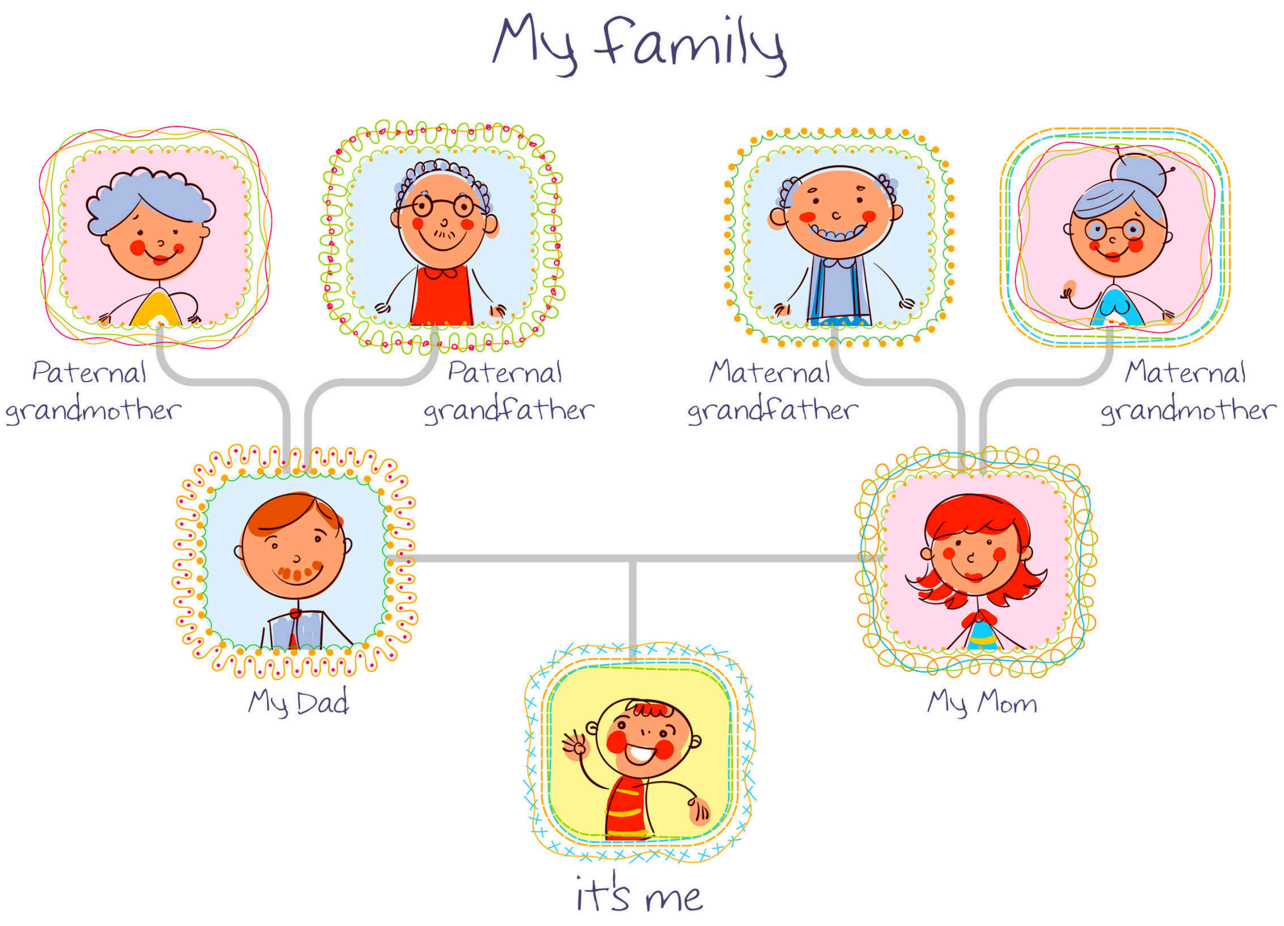
Trees are not useful only for efficient search. They are useful for representing hierarchies and relationships in general.



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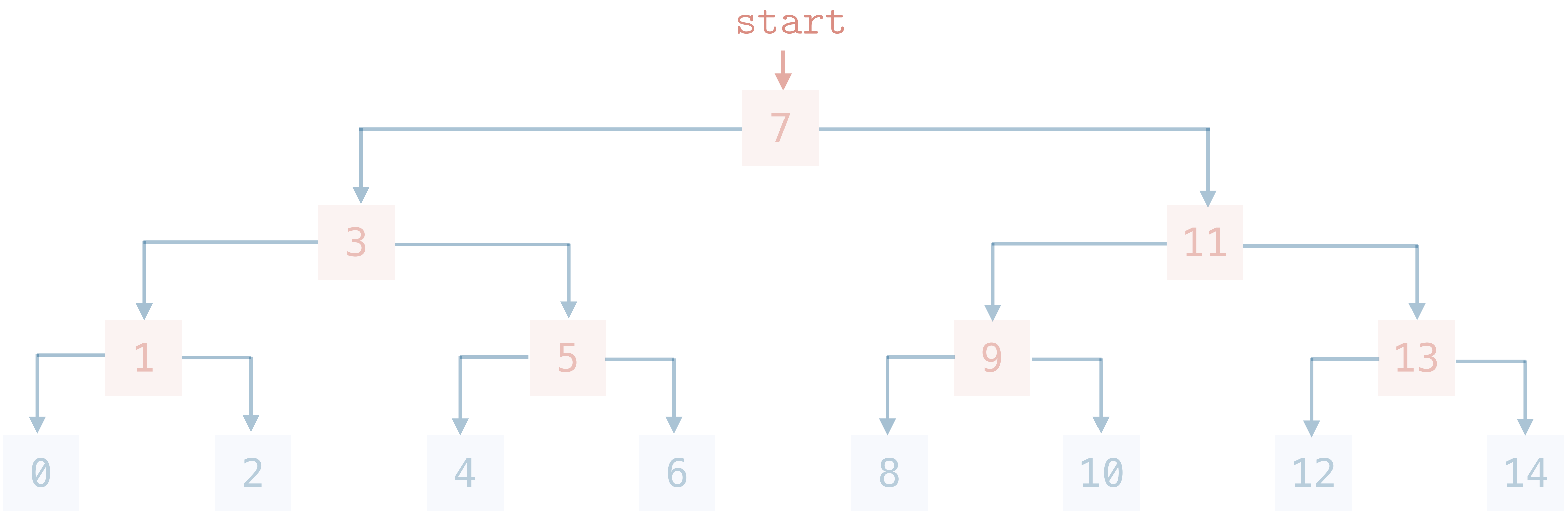

A Binary Search Tree!

We need to build a linked *tree structure.*

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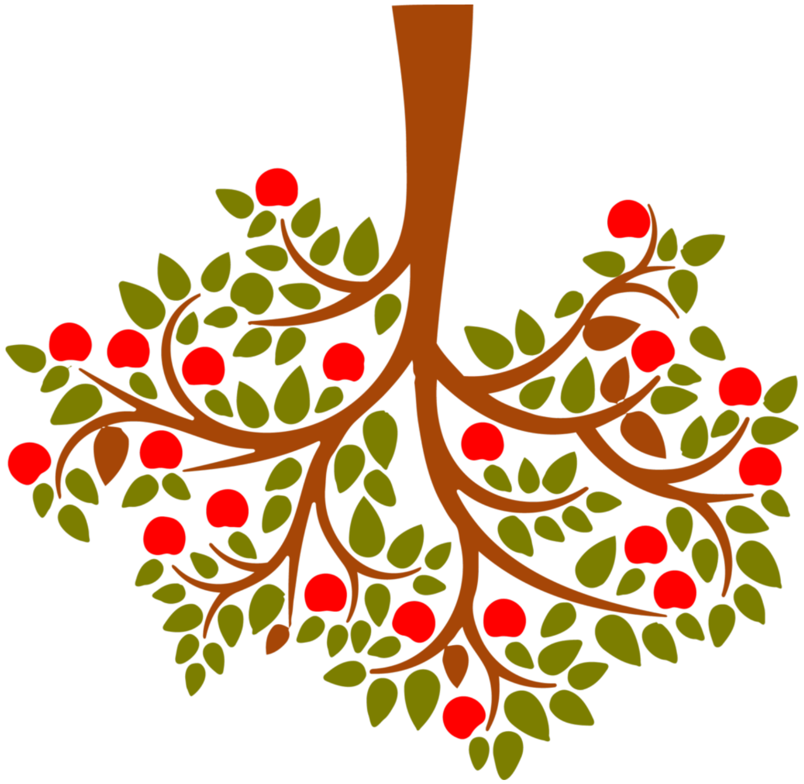
Questions.

- How do we *insert* into the tree?
- How do we *remove* from the tree?
- How do we *search* the tree?
- How do we *traverse* the tree?
- What *other operations* can we perform?
- What *properties* do trees have?
- … ?



Tree Data Structures

- Definitions and properties

Operations on BSTs

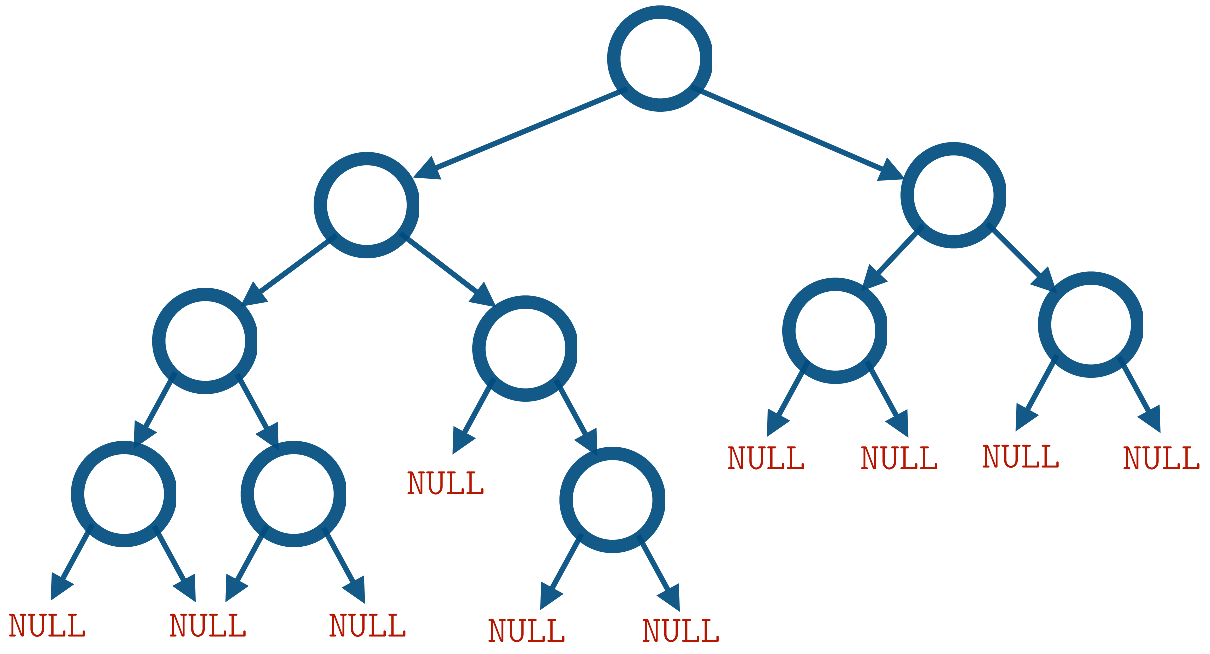
Balanced binary search trees

Tree traversals

Terminology

Definition. A *tree* is:

- `NULL` or
- a node connected to a set of *disjoint* trees.



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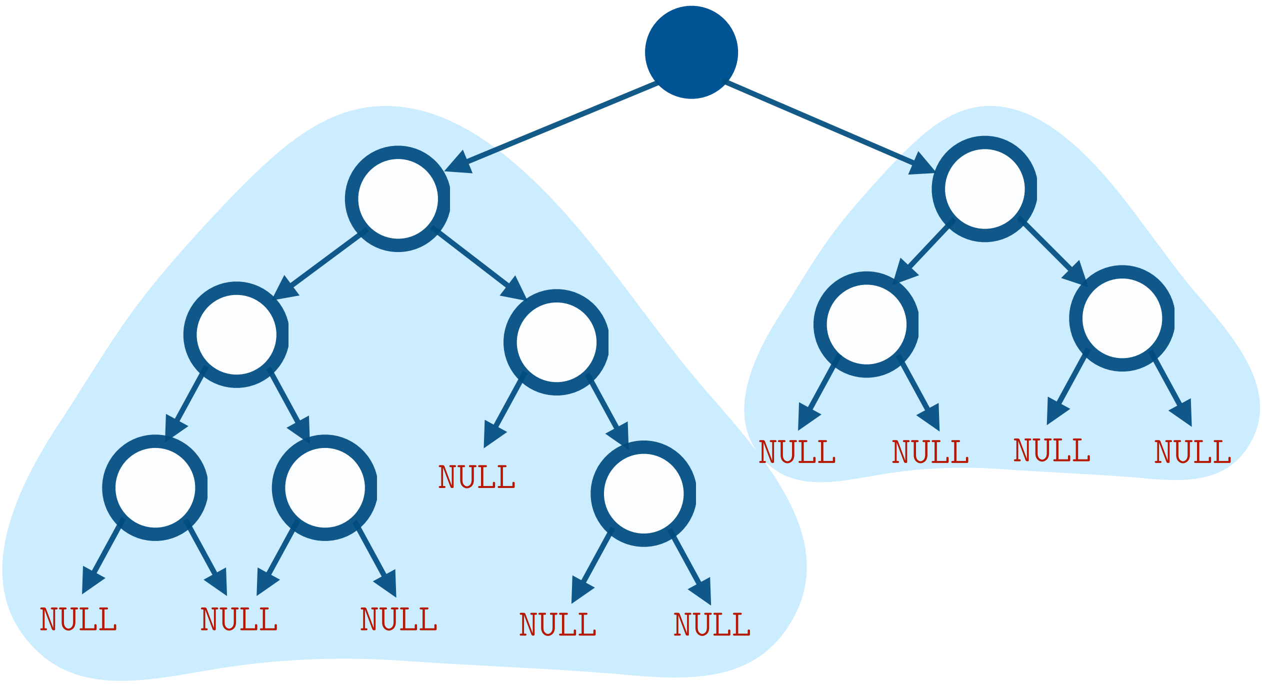

A node connected to two disjoint trees

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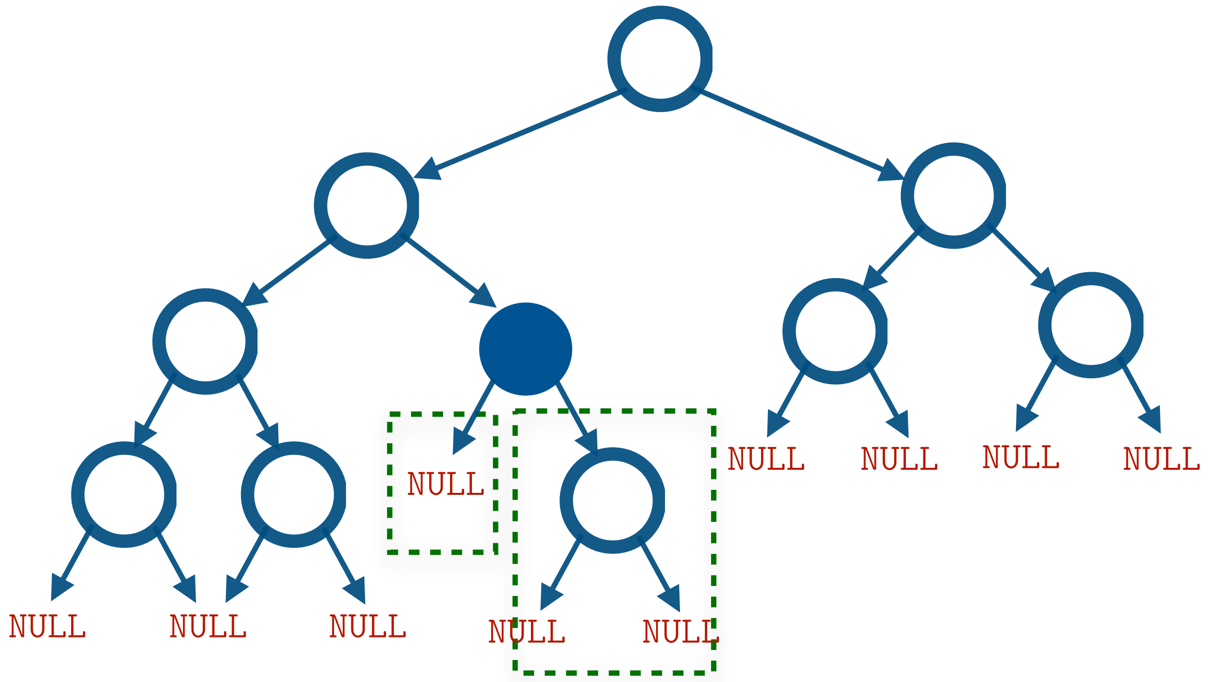

A node connected to two disjoint trees:

- A NULL tree and
- A node connected to two NULL trees

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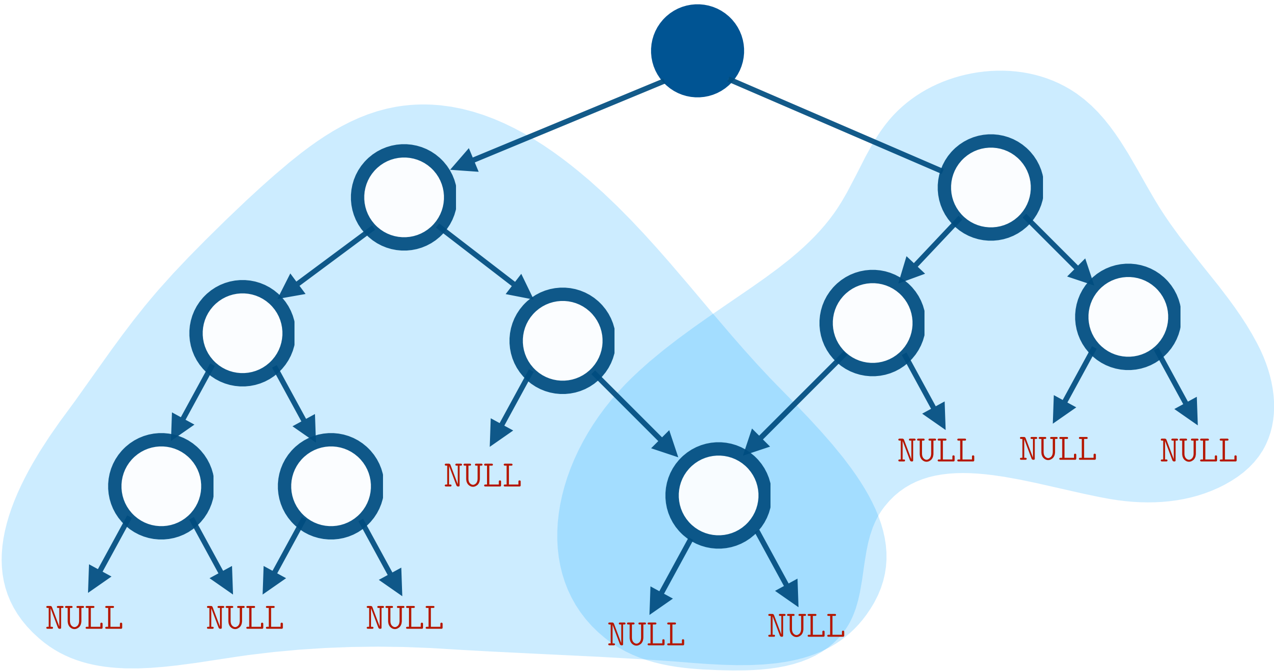

not a tree: A node connected to two overlapping trees

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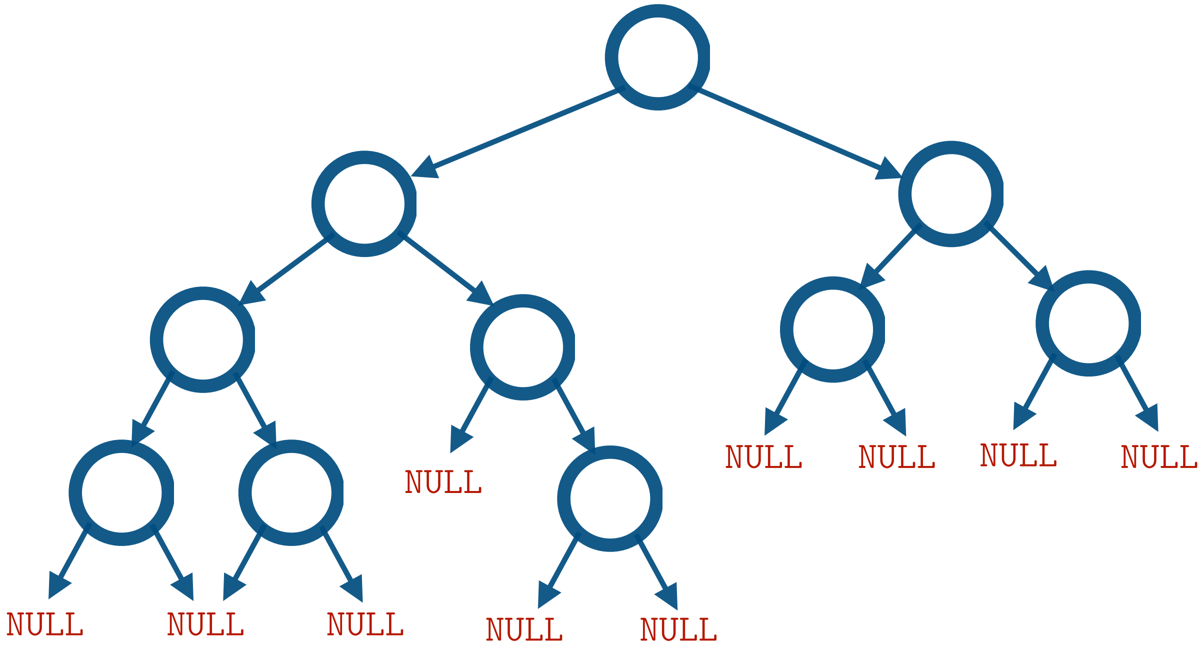
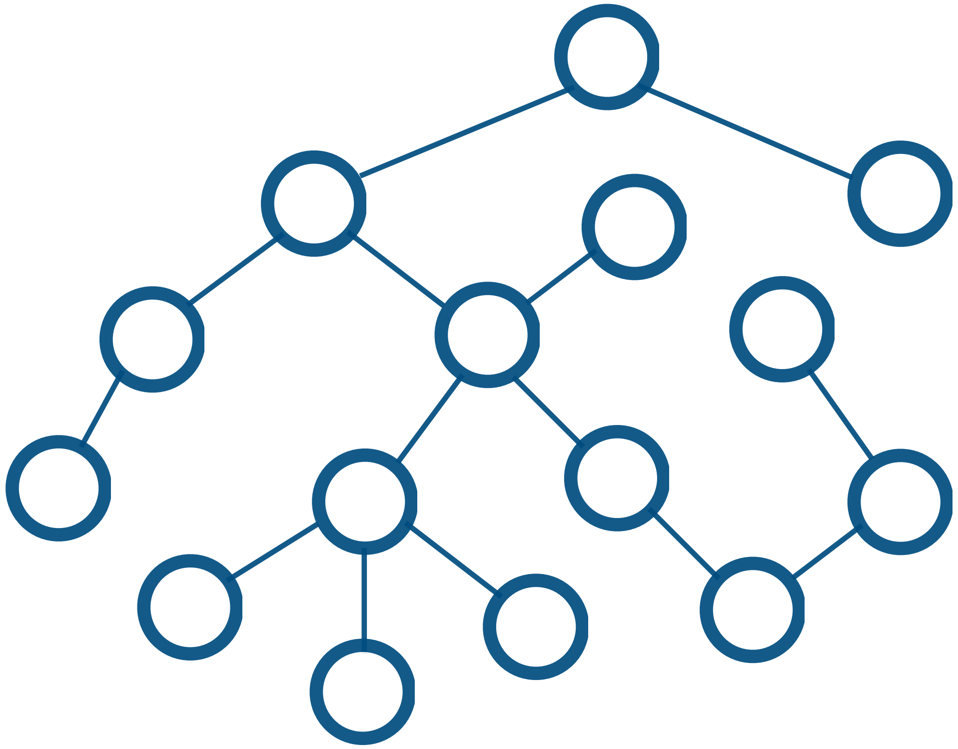


Alternative definition (from graph theory).
A *tree* is a set of nodes in which any two nodes are connected by exactly one path



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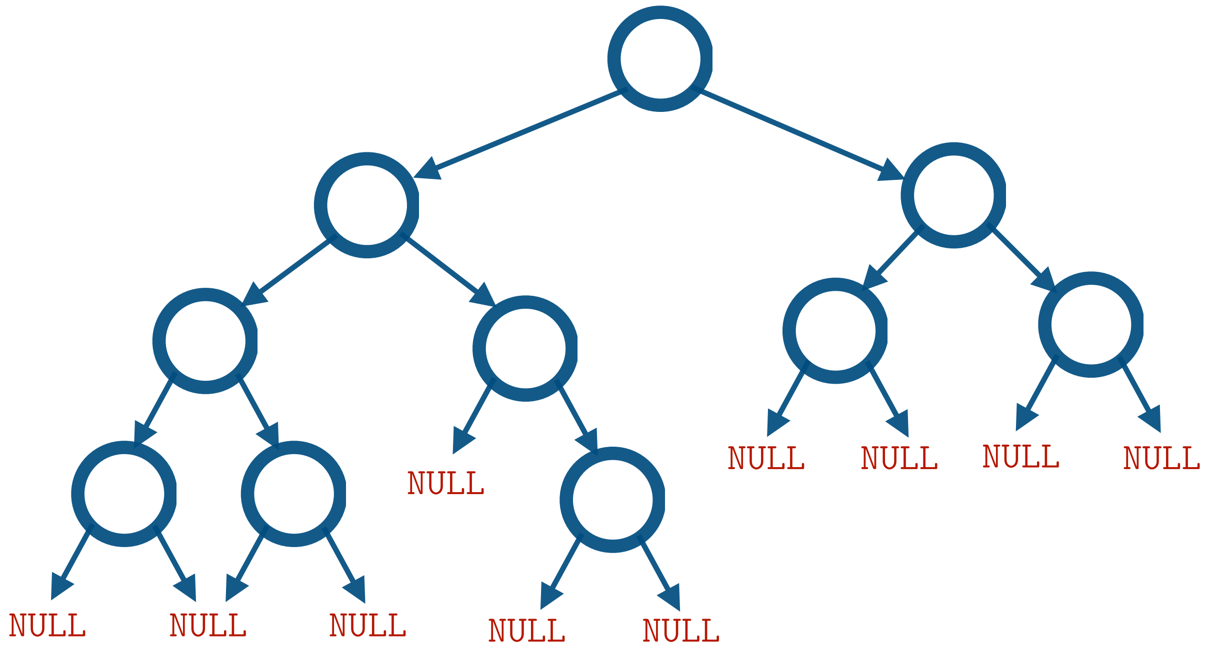

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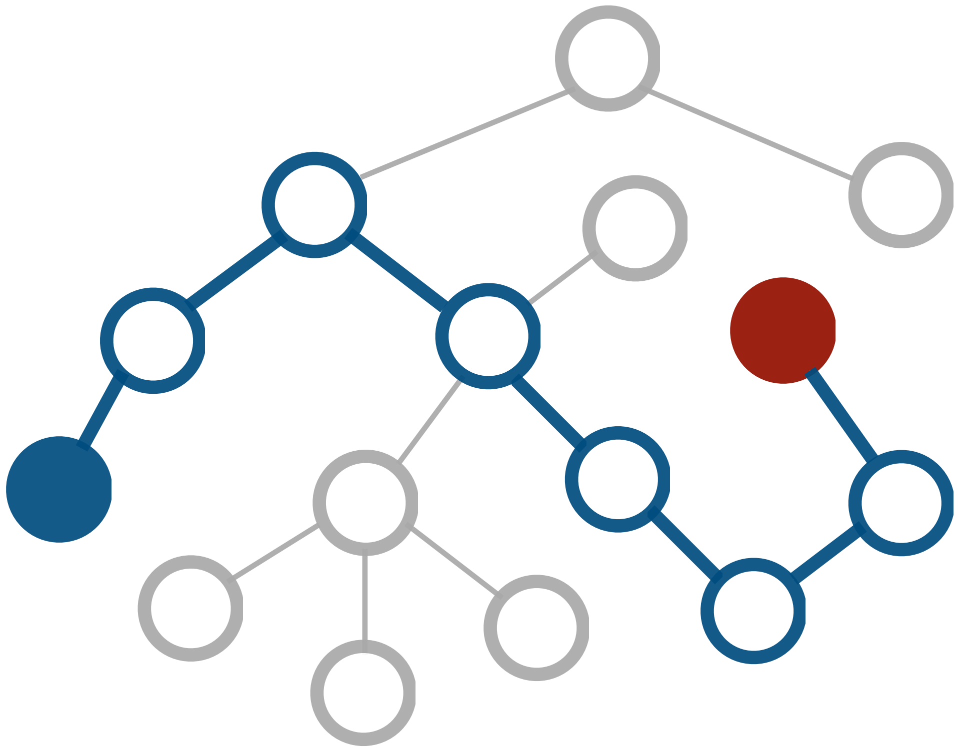

only one path from the red to the blue node

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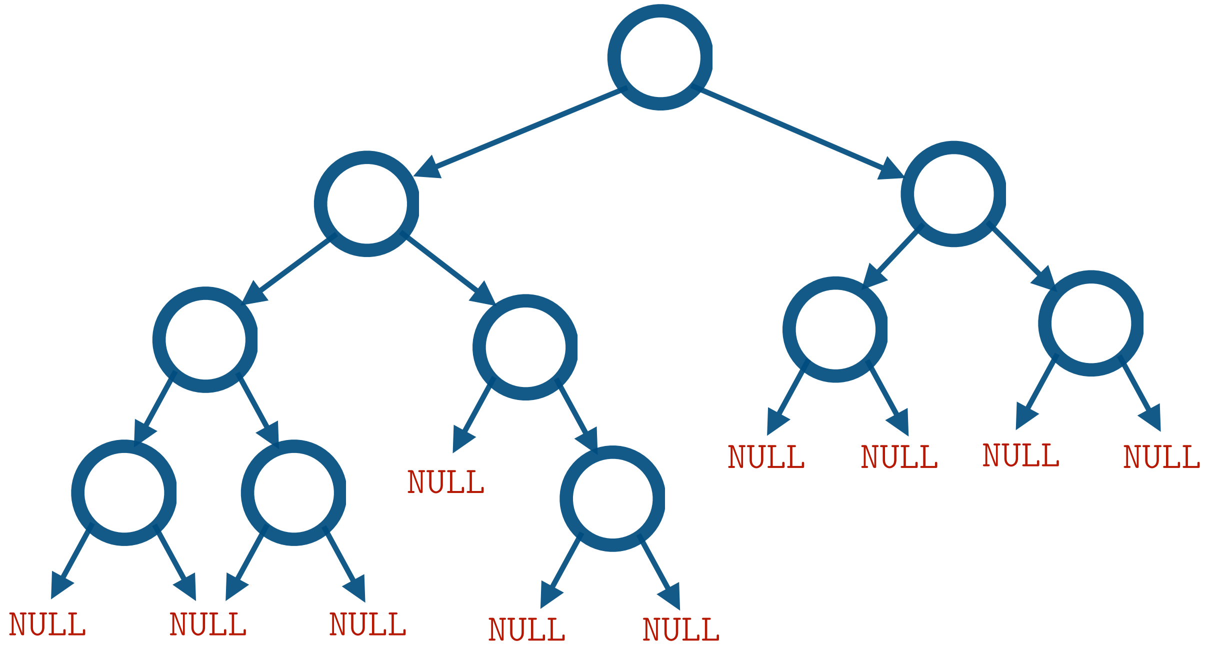

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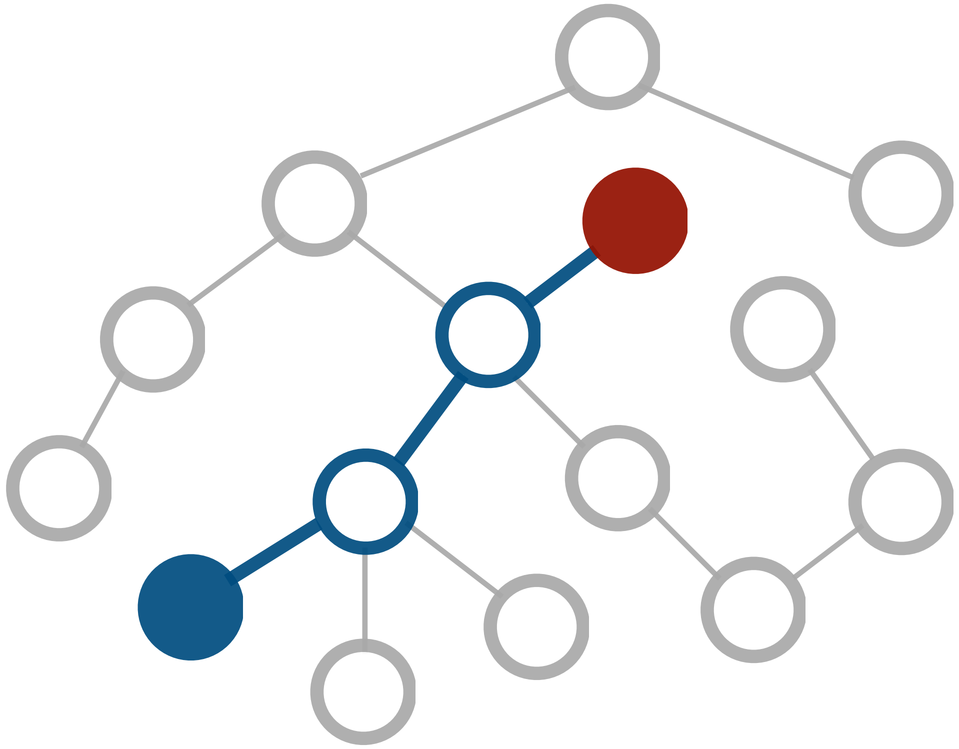

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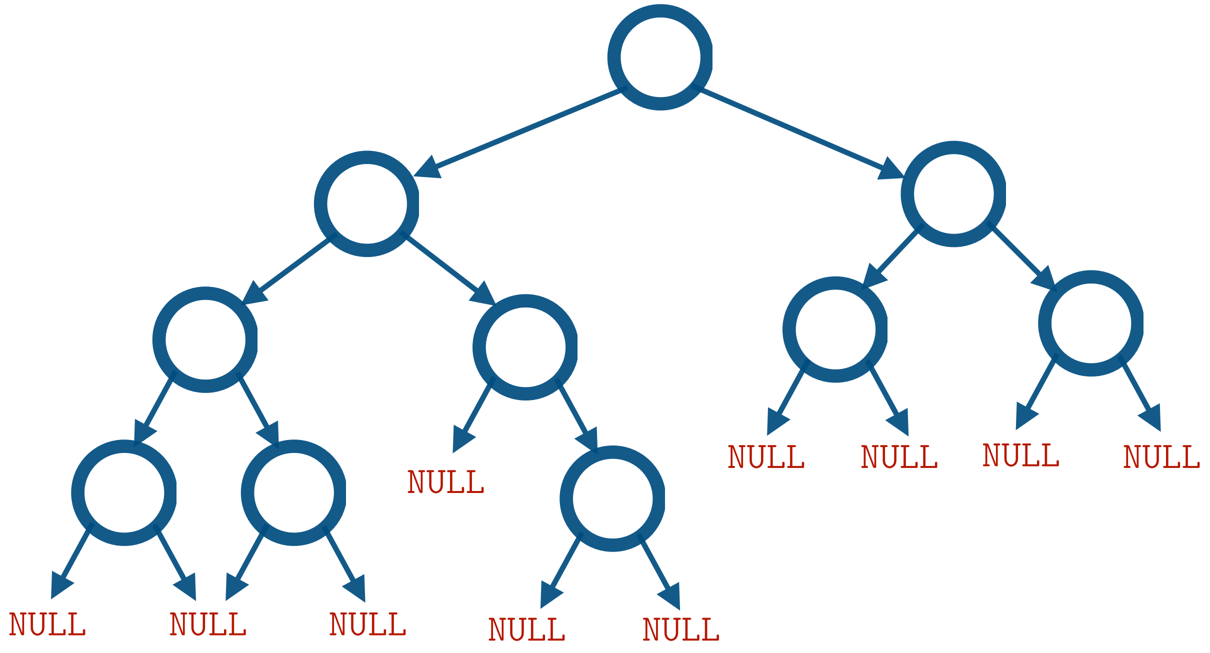

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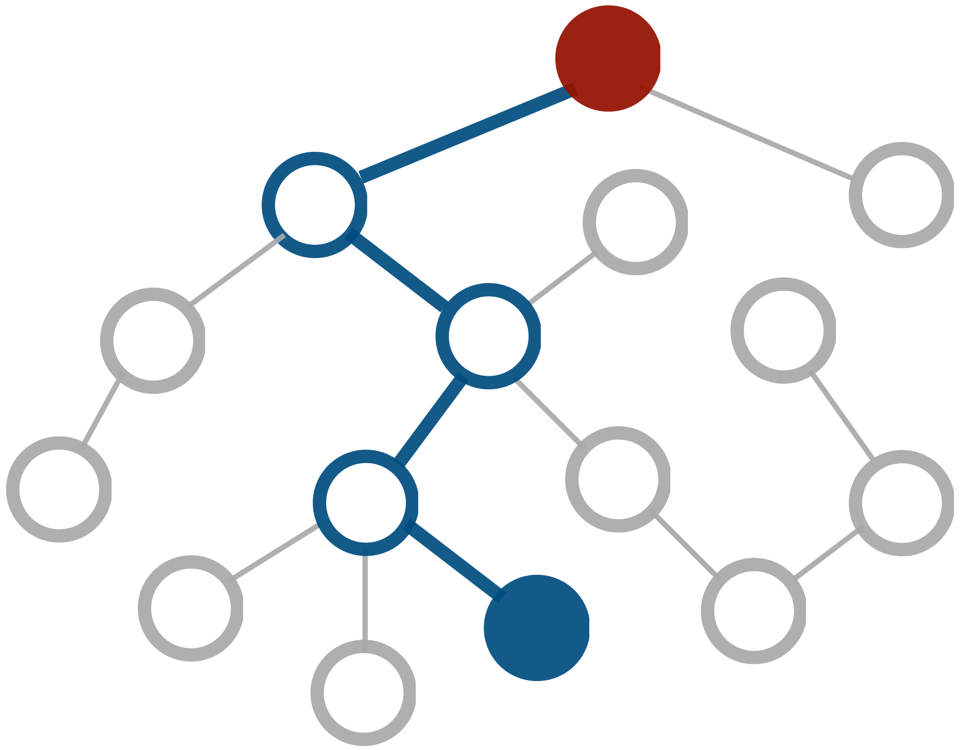

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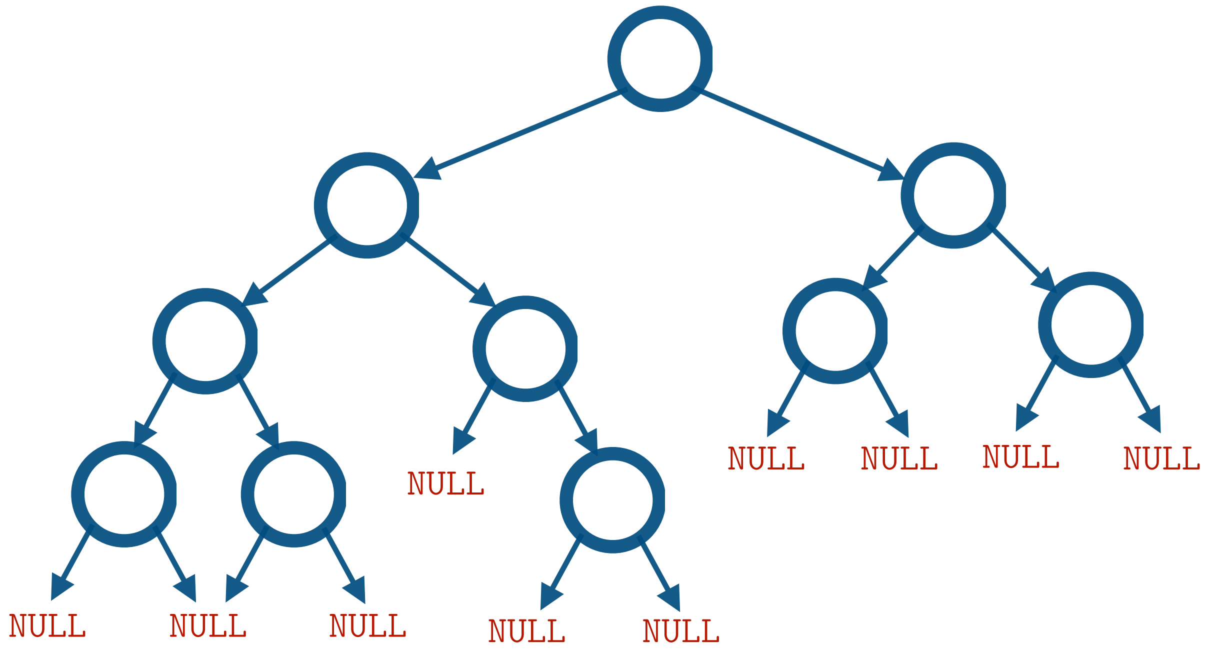

A binary tree is:

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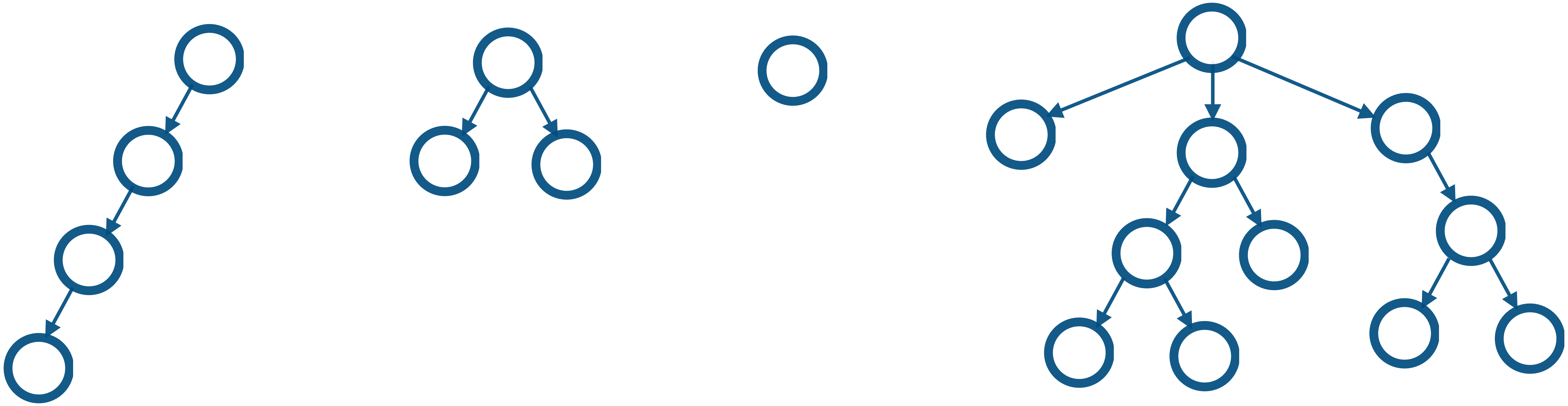

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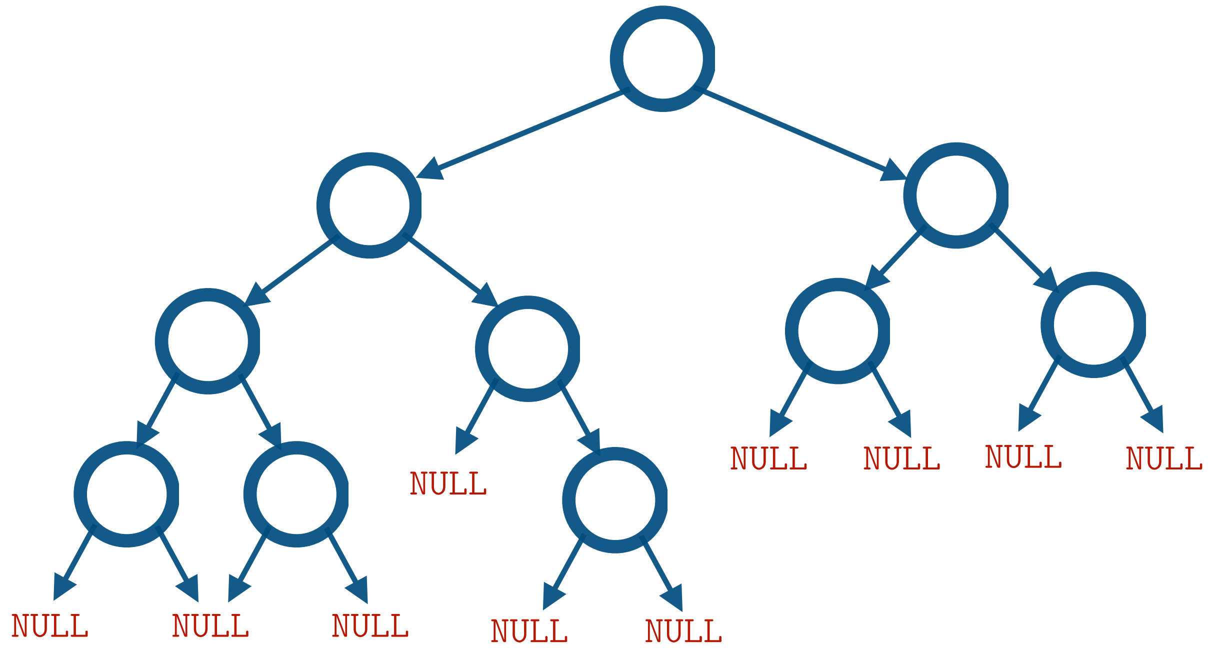

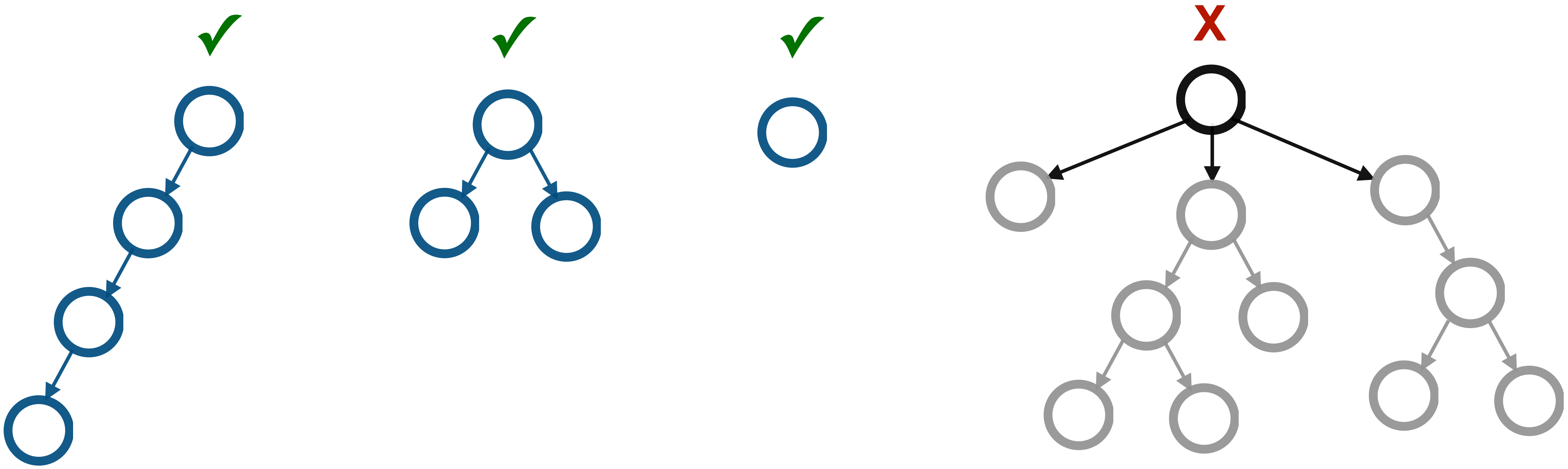

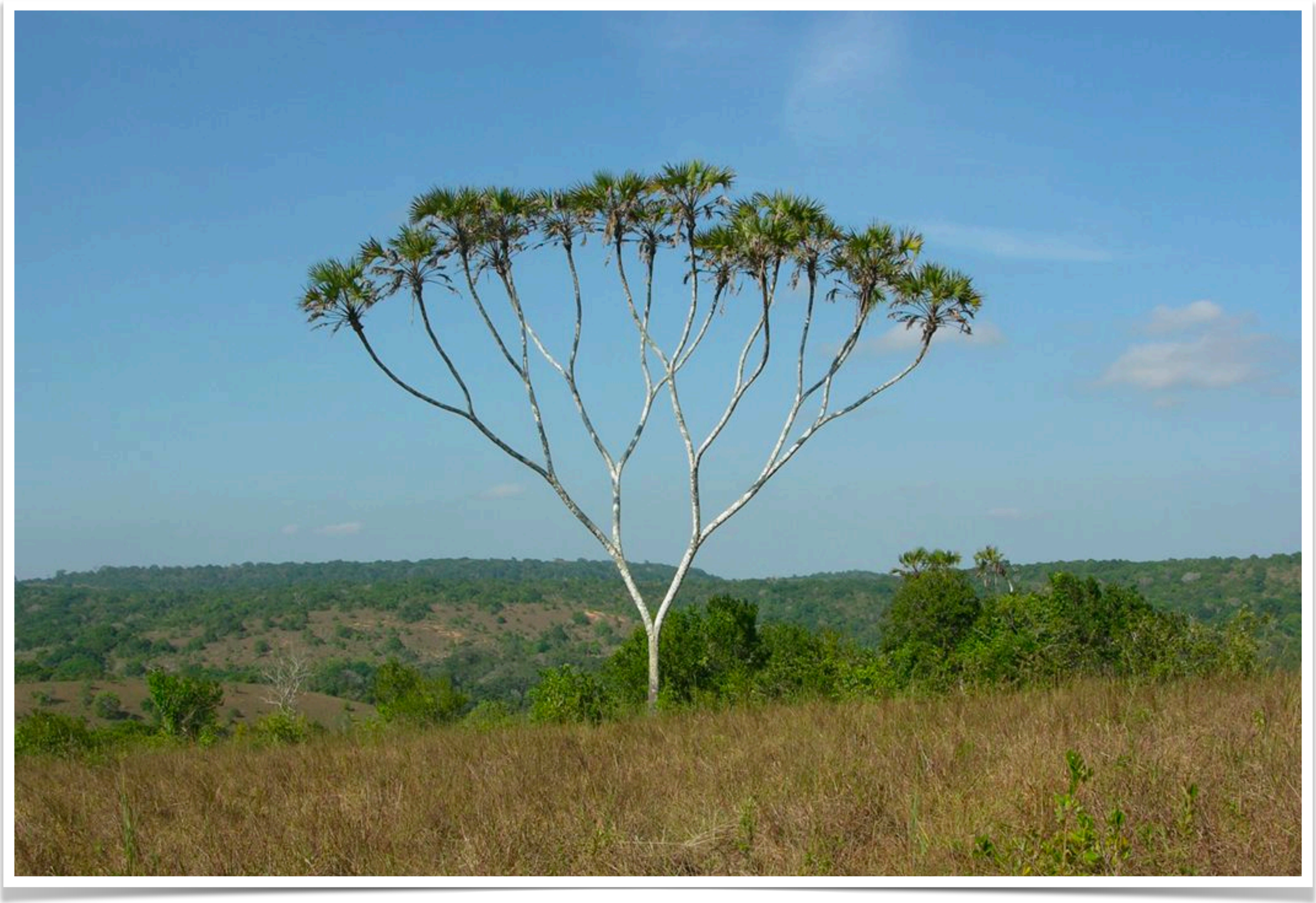

A binary tree in nature!

Binary trees?

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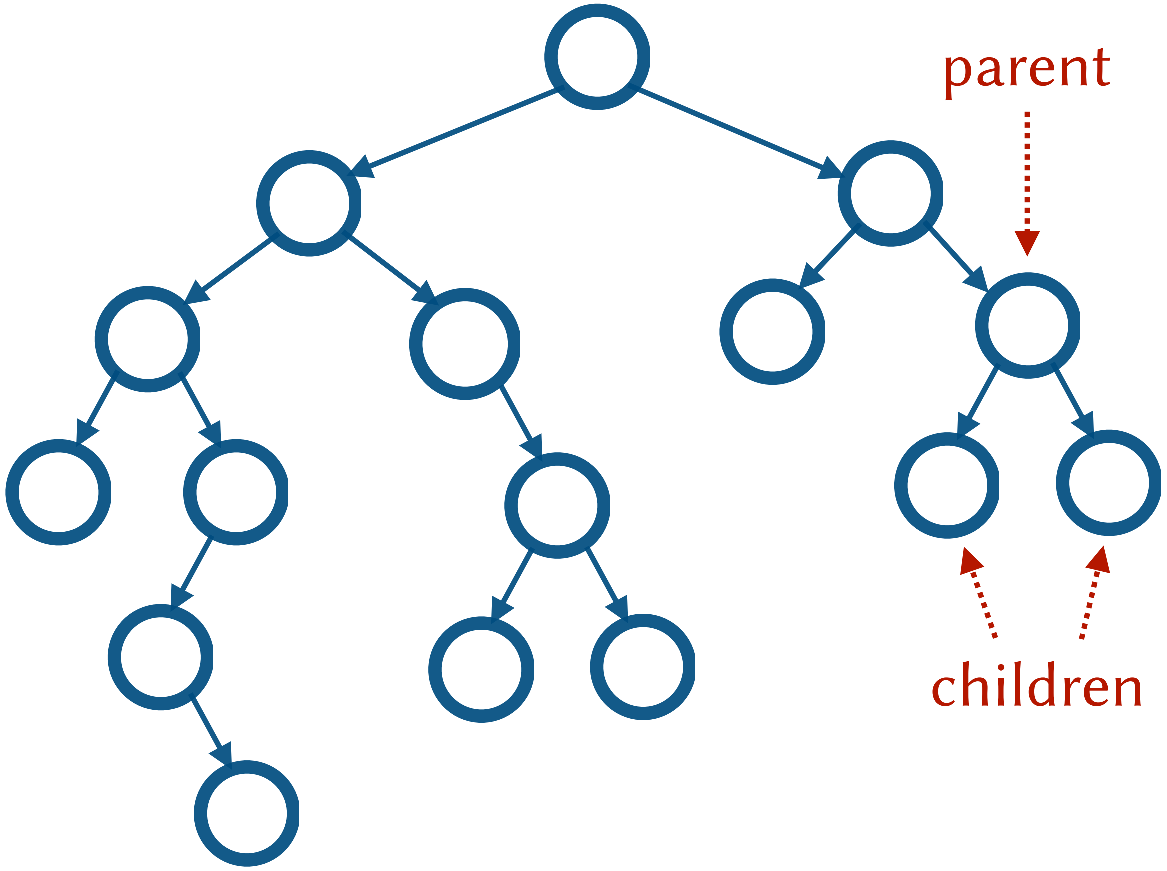

nodes that share the same parent are siblings

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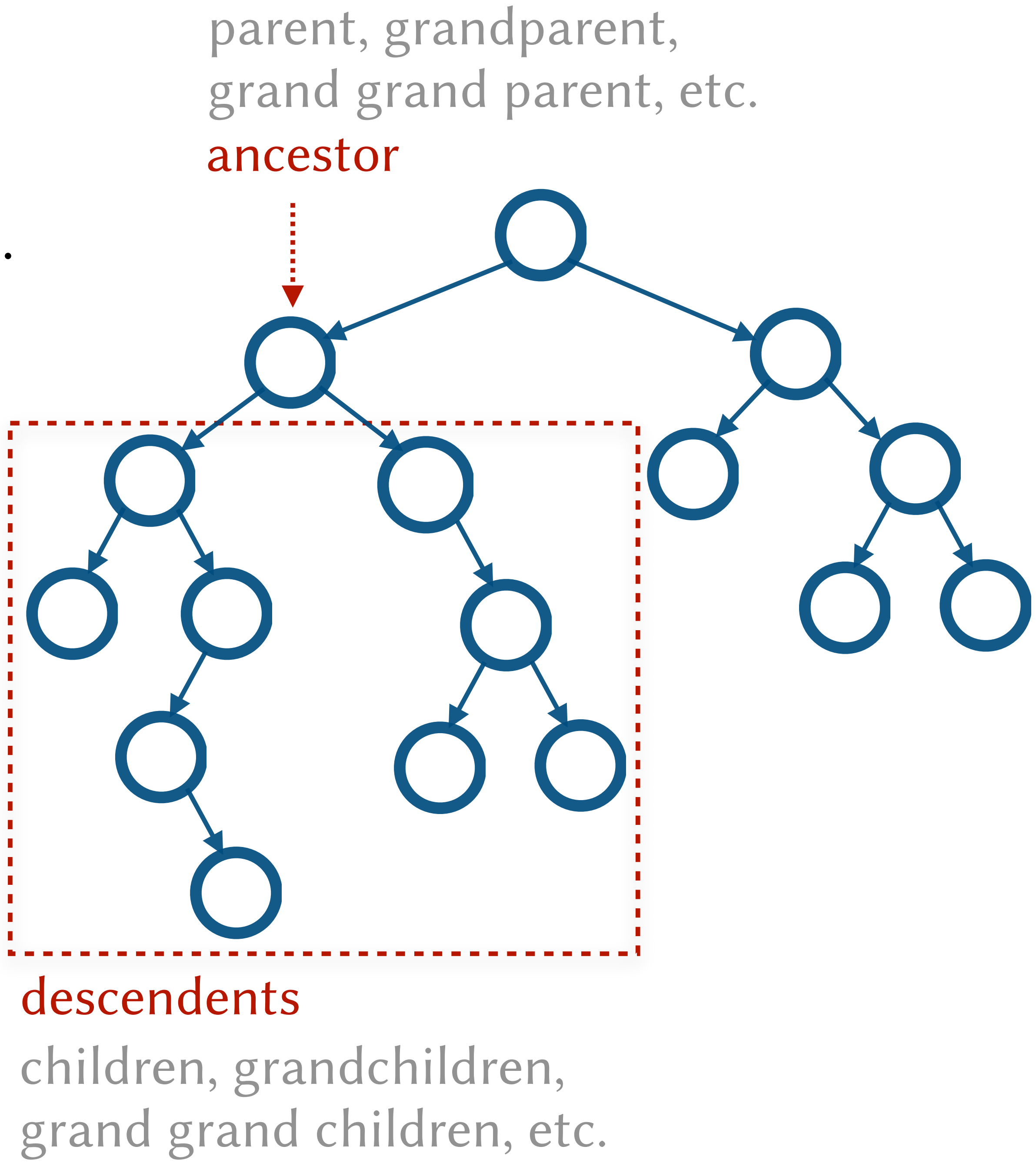

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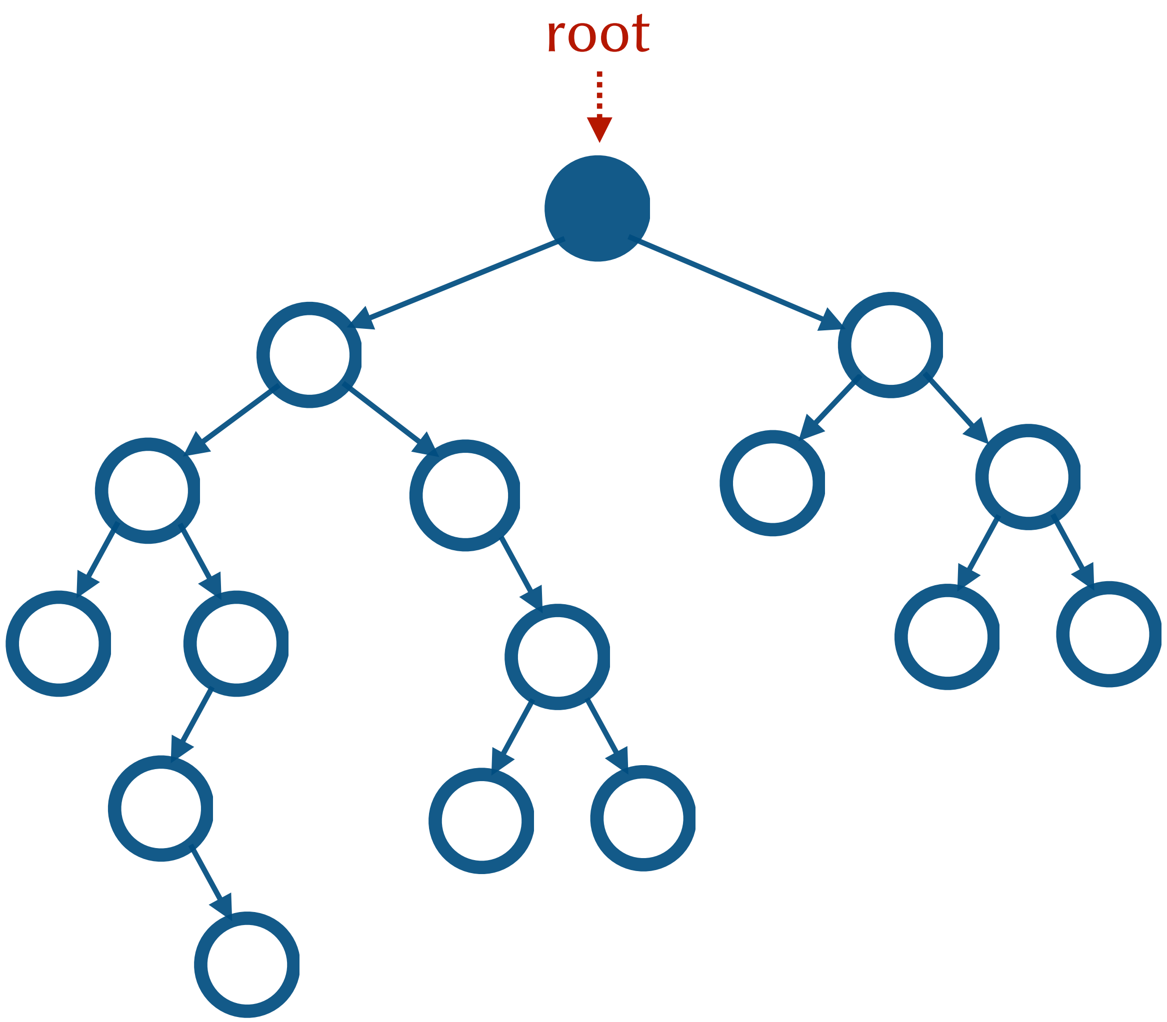
Every tree has one root node.

The root node has no parent.
The root is an ancestor for all nodes.
All nodes are descendants of the root.



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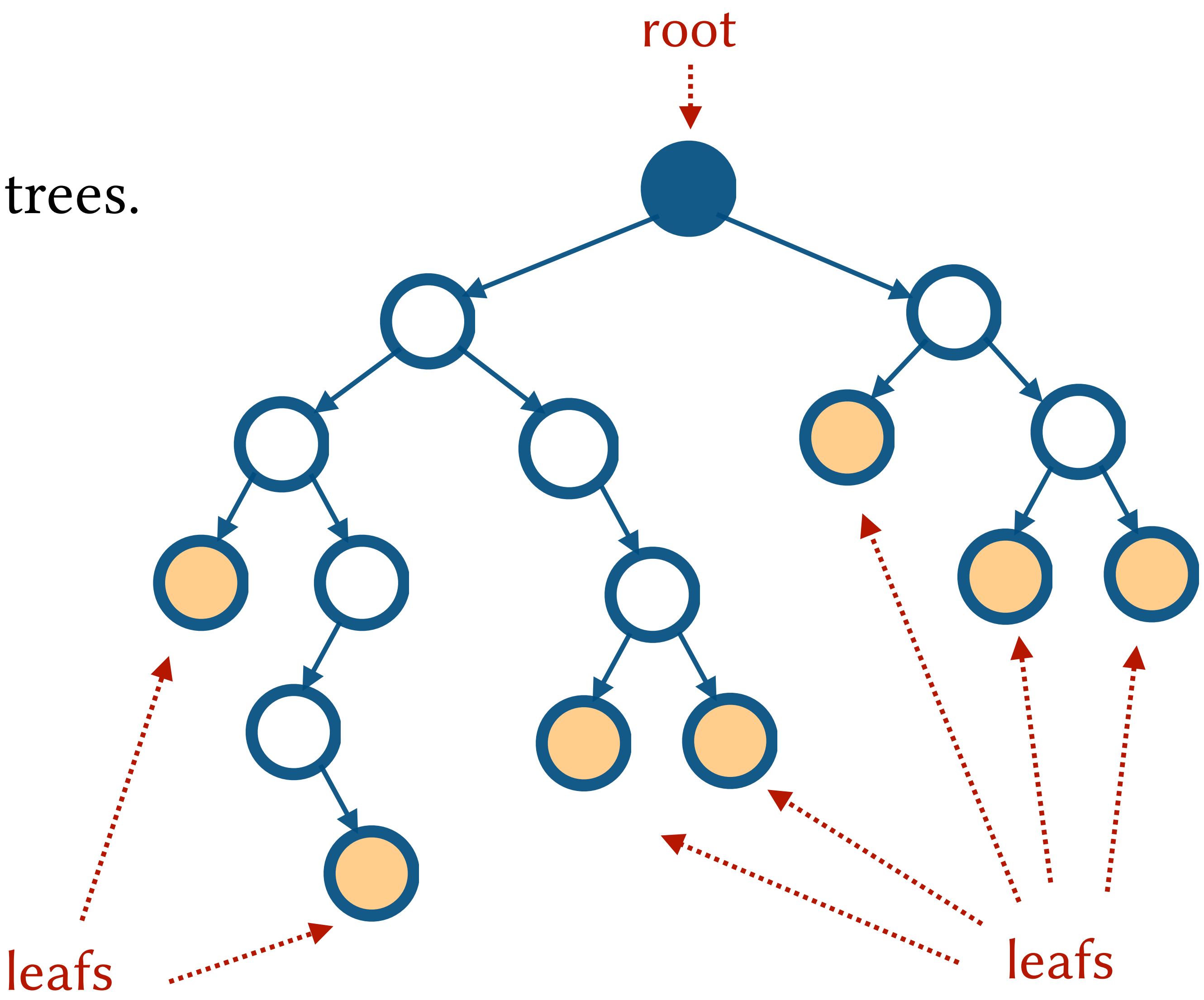

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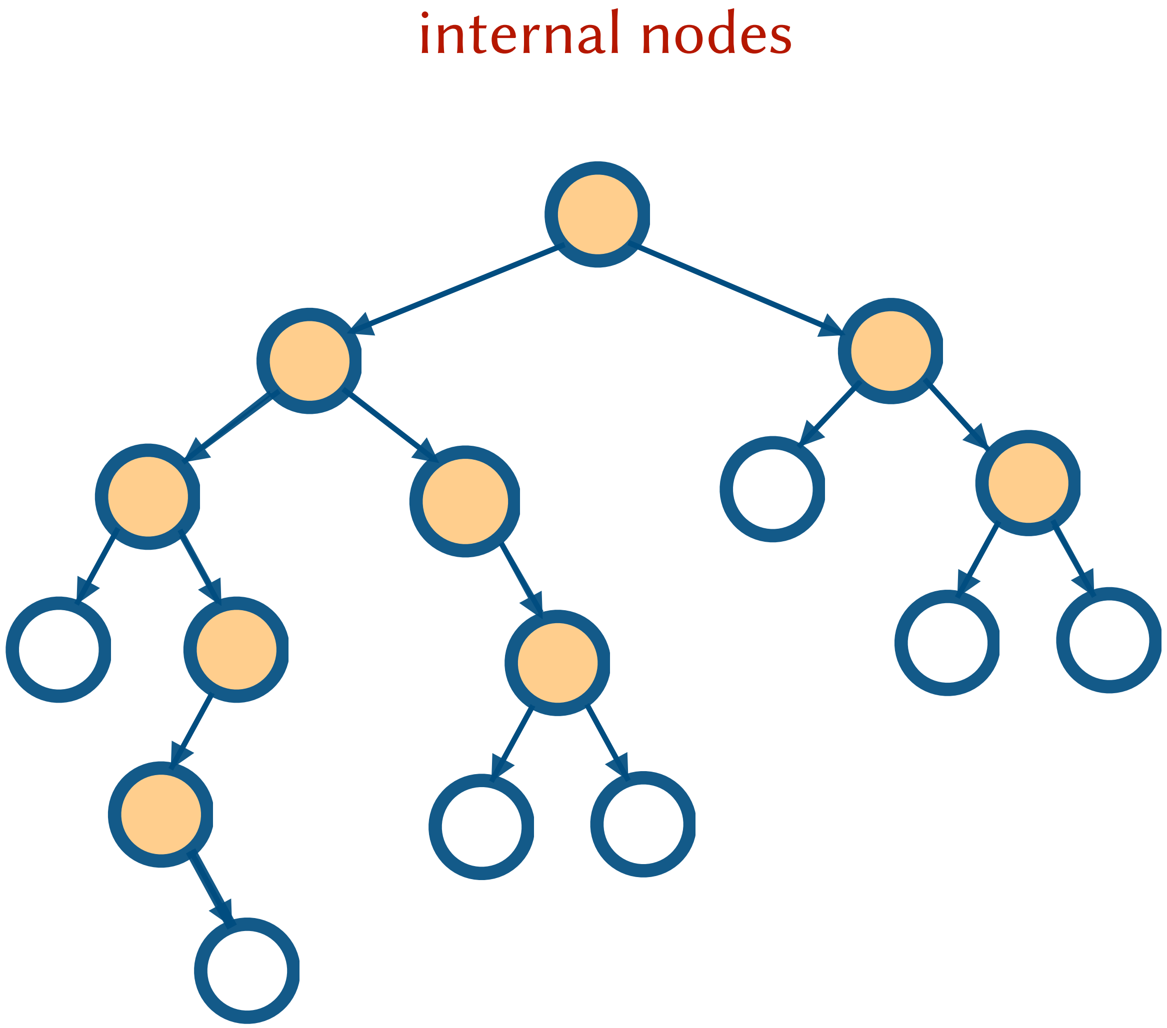
Leafs are also called "external nodes".



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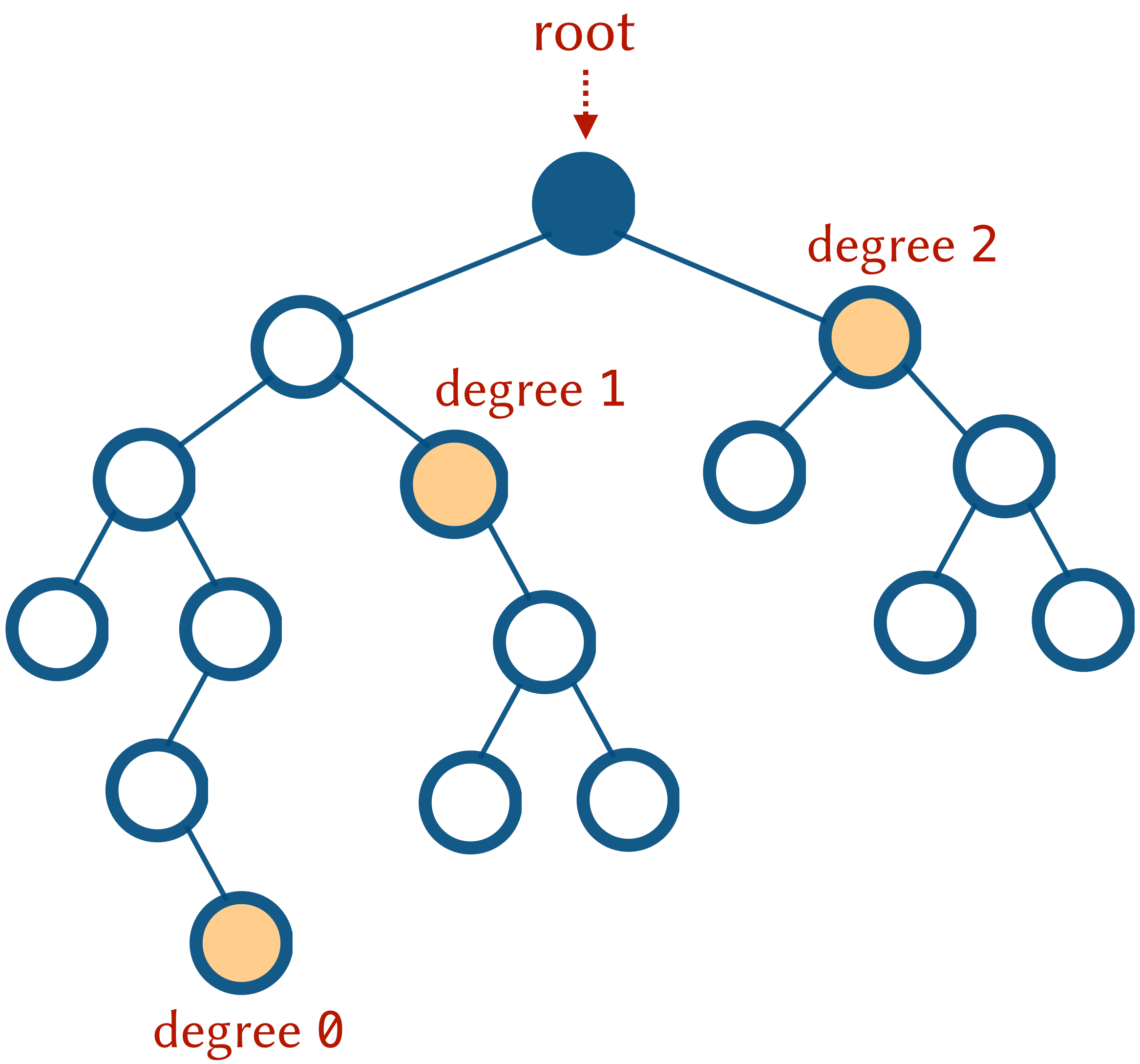
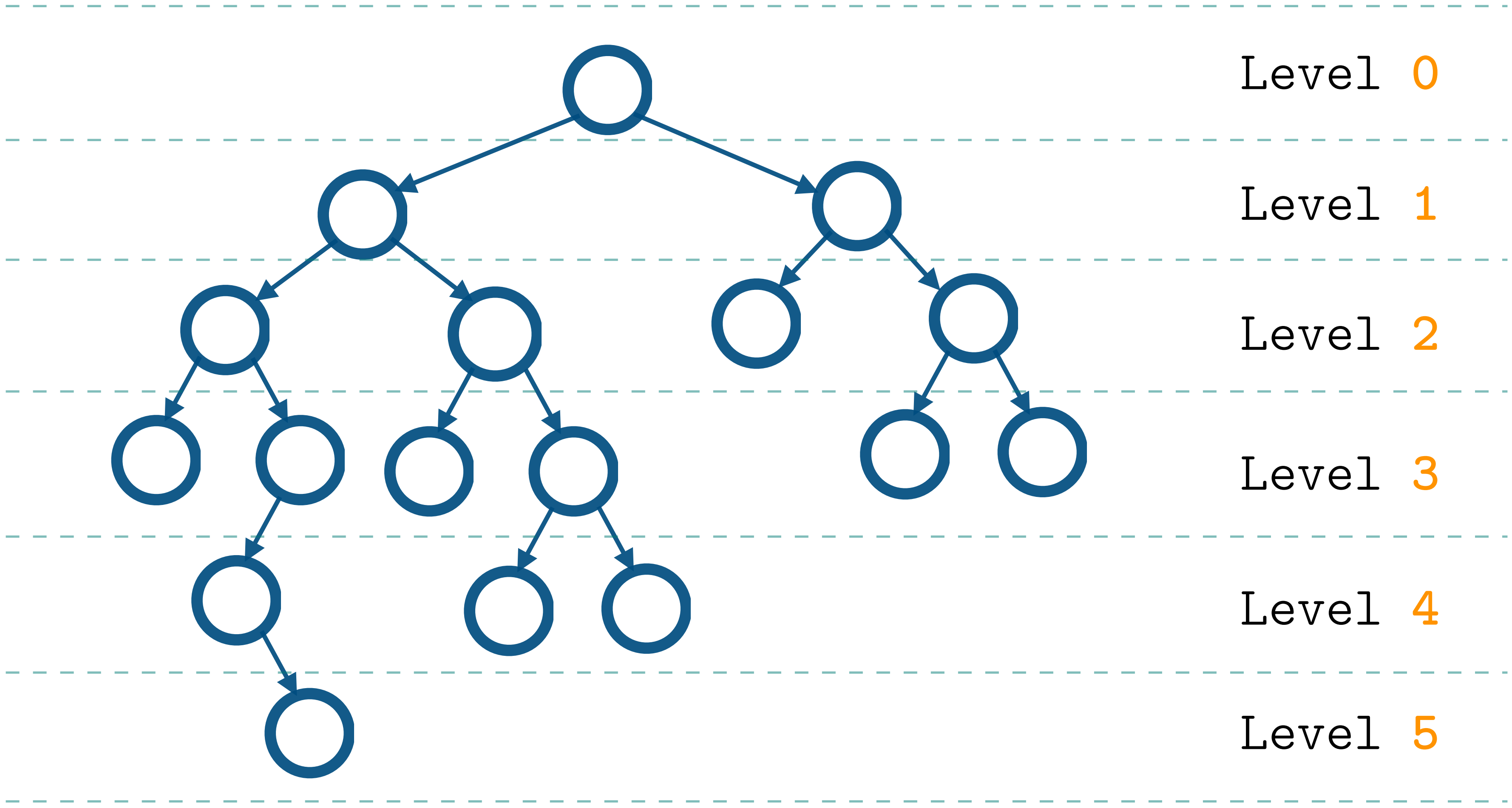
The degree of a node is the number of its children.
The degree of a tree is the maximum degree of a node in the tree.

A leaf has degree 0. The illustrated tree has degree 2.



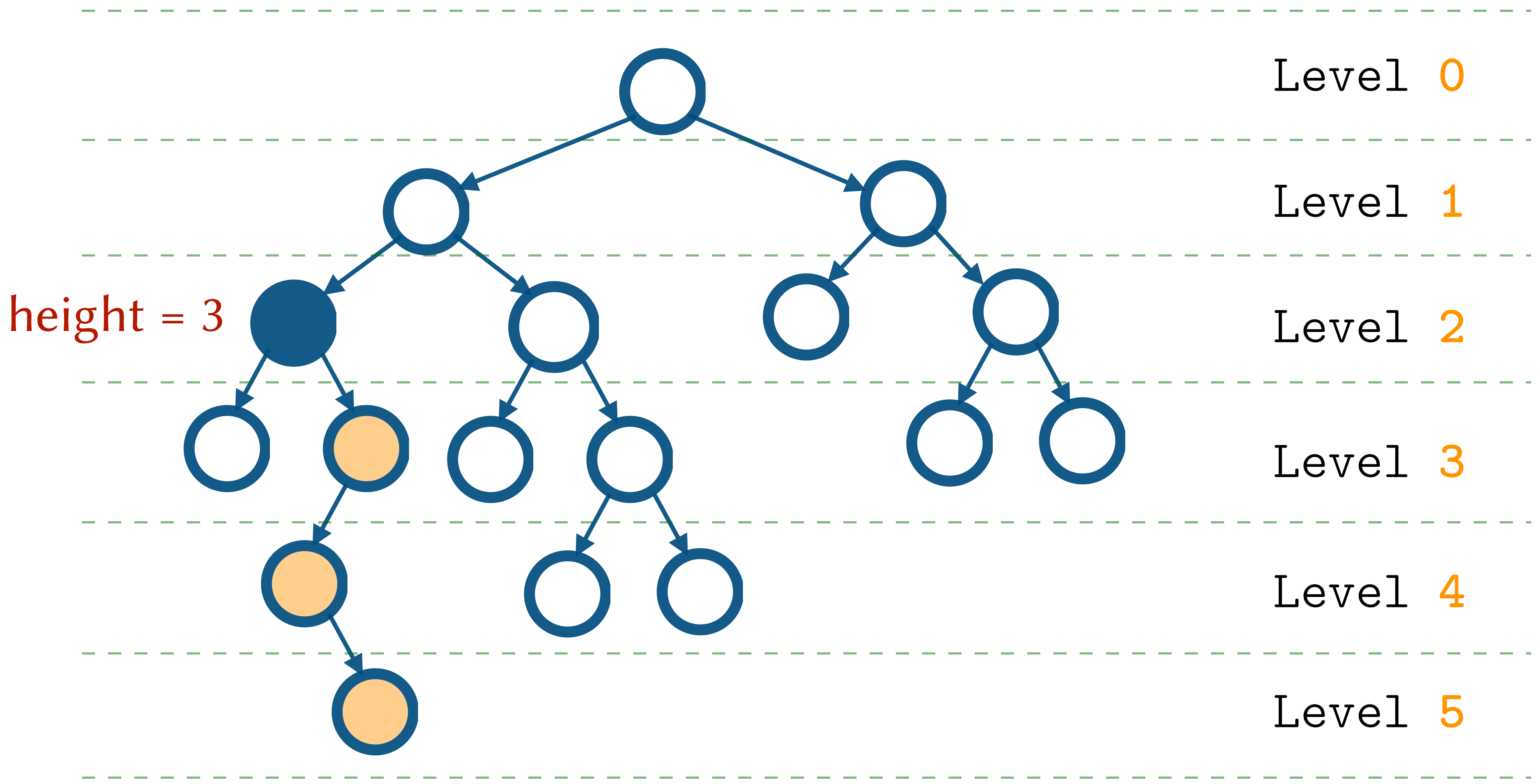
Terminology



The depth of a node is its level in the tree.

The root has depth 0.

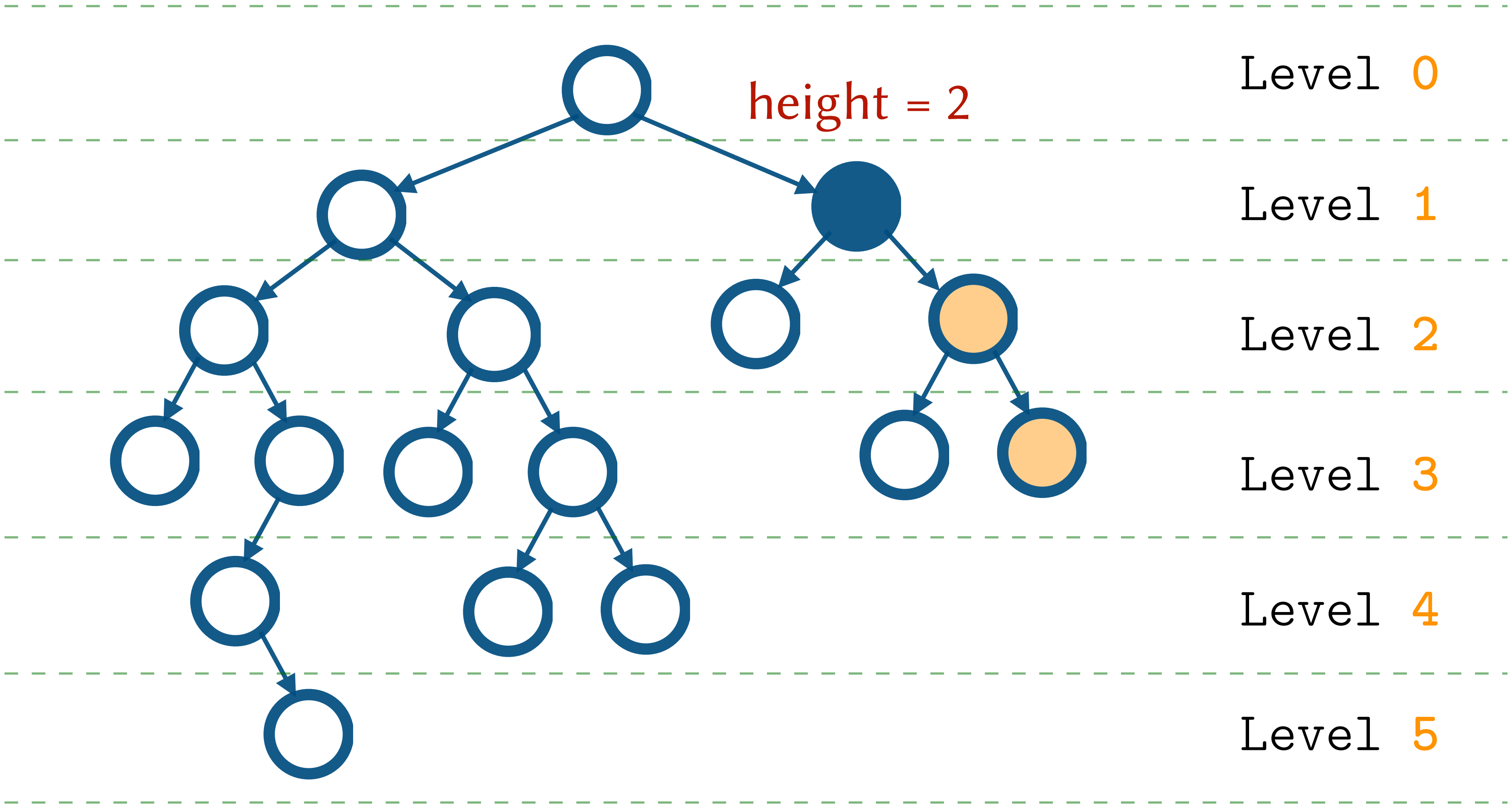
Terminology



The depth of a node is its level in the tree.
The root has depth 0.

The height of a node is the maximum number of levels below it.
All leafs have height 0.

Terminology



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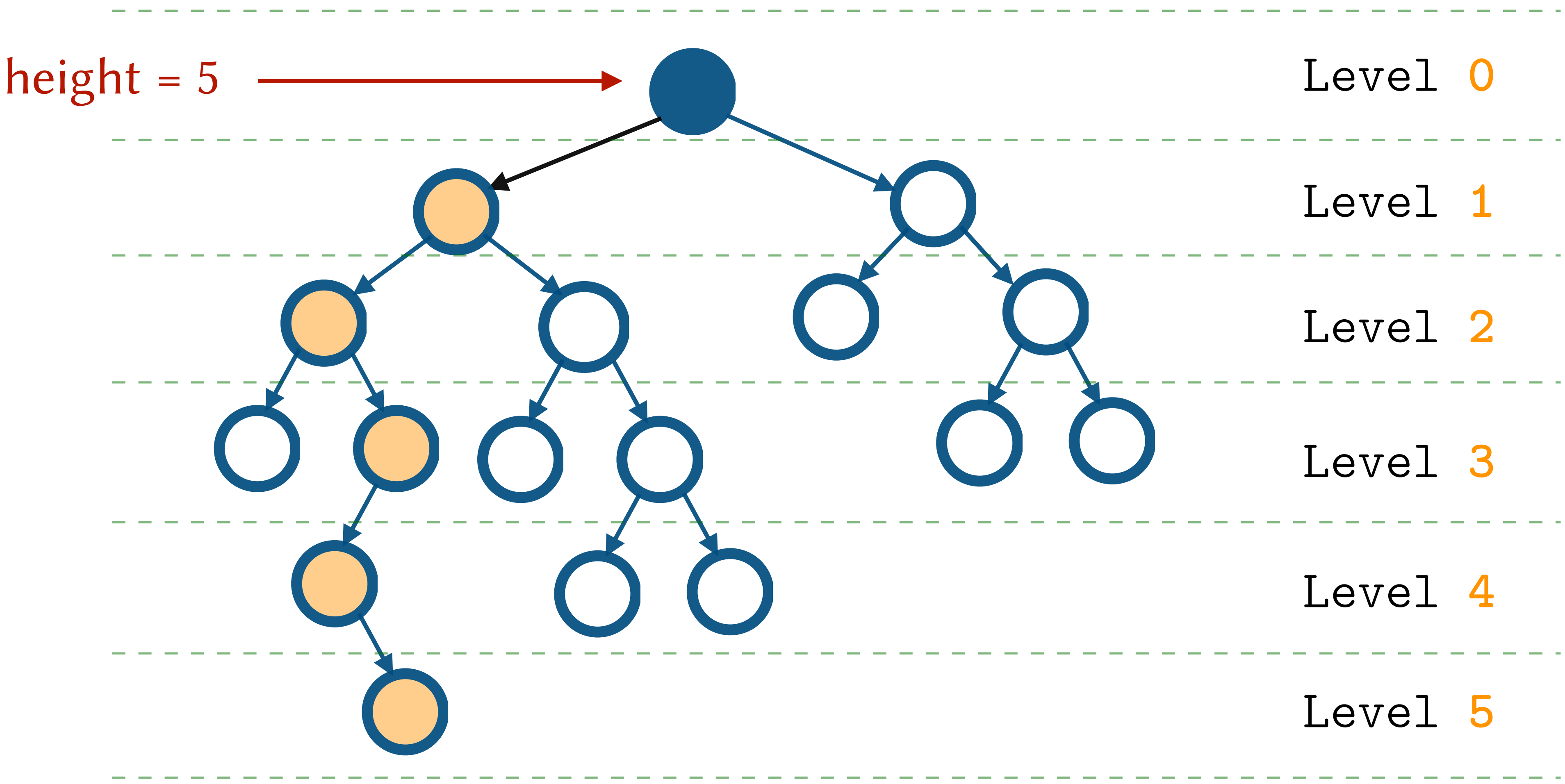

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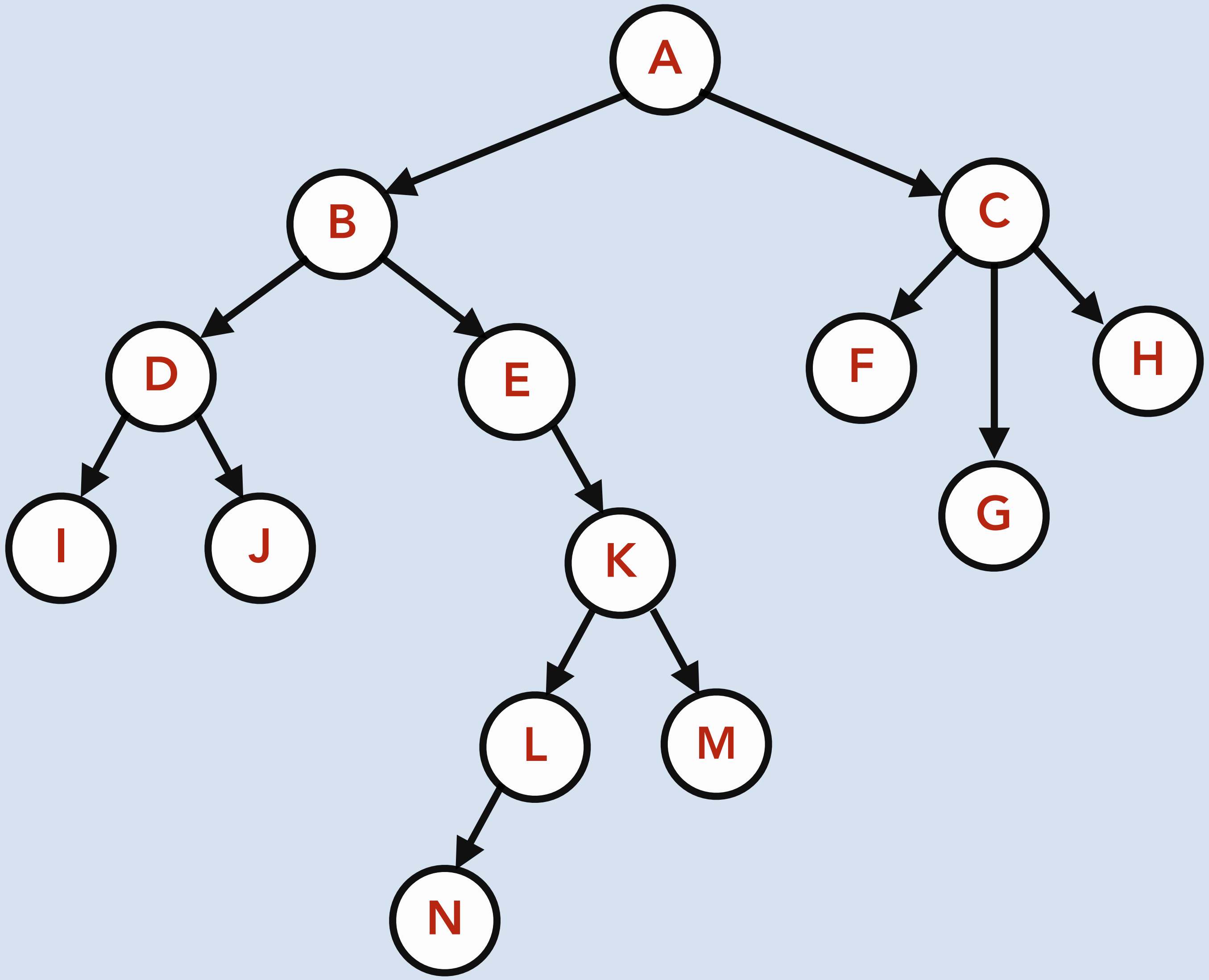
All leafs have height 0.

The height of a tree is the height of the root.

Also called "tree depth".

Exercise



Root =

Leafs =

Tree height =

Tree Degree =

Ancestors of **L** =

Height of **E** =

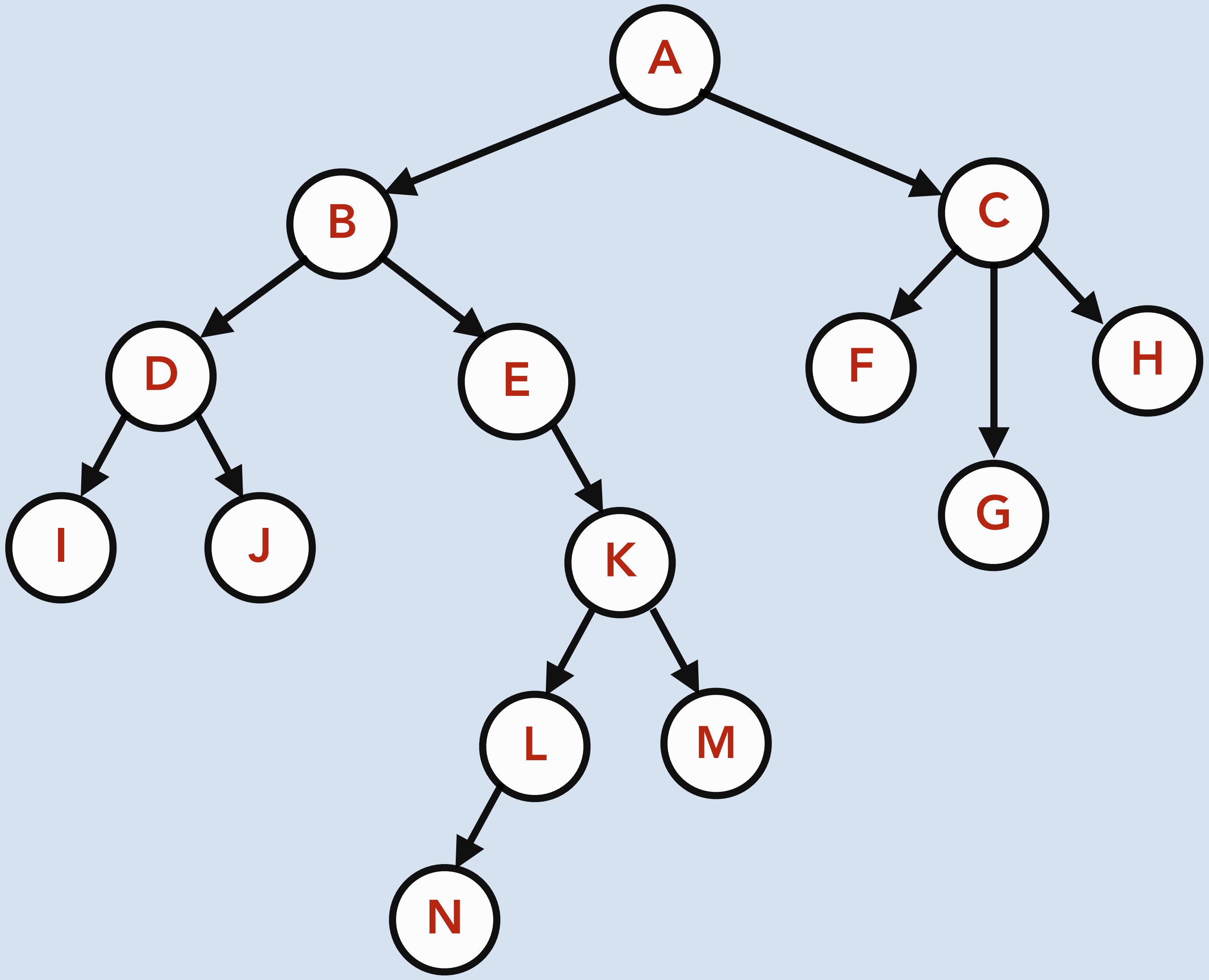
Depth of **E** =

Degree of **E** =

Descendants of **E** =

Siblings of **G** =

Exercise



Root = **A**

Leafs = **I J N M F G H**

Tree height = **5**

Tree Degree = **3**

Ancestors of **L** = **K E B A**

Height of **E** = **3**

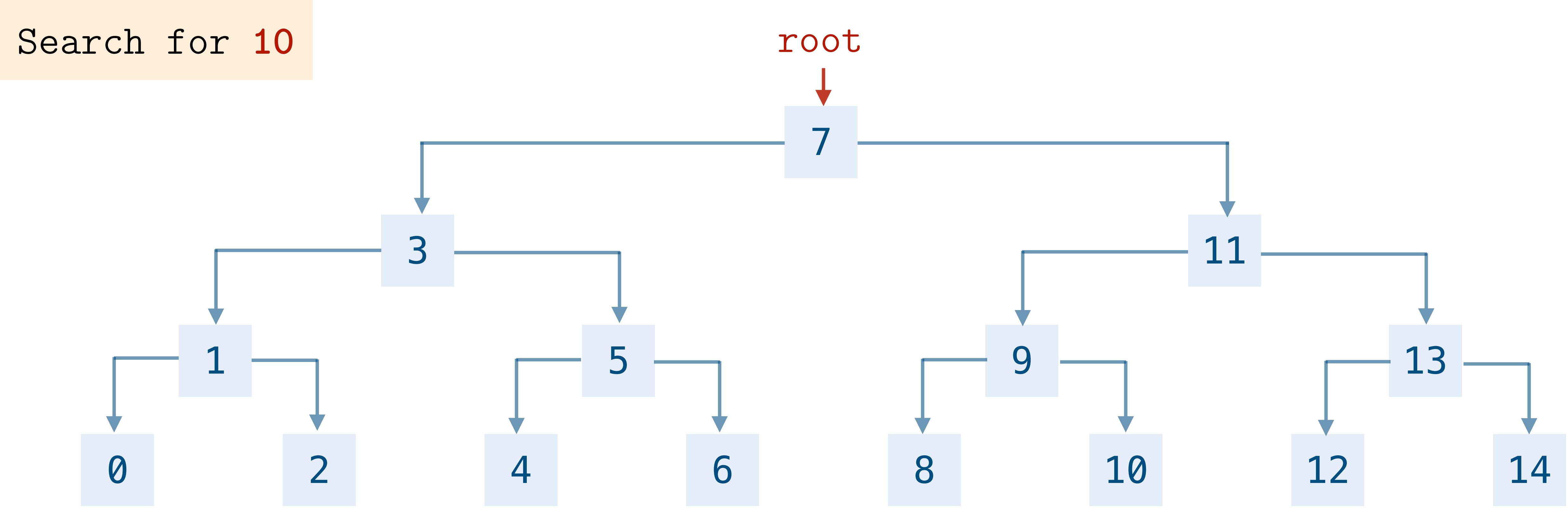
Depth of **E** = **2**

Degree of **E** = **1**

Descendants of **E** = **K L M N**

Siblings of **G** = **F H**

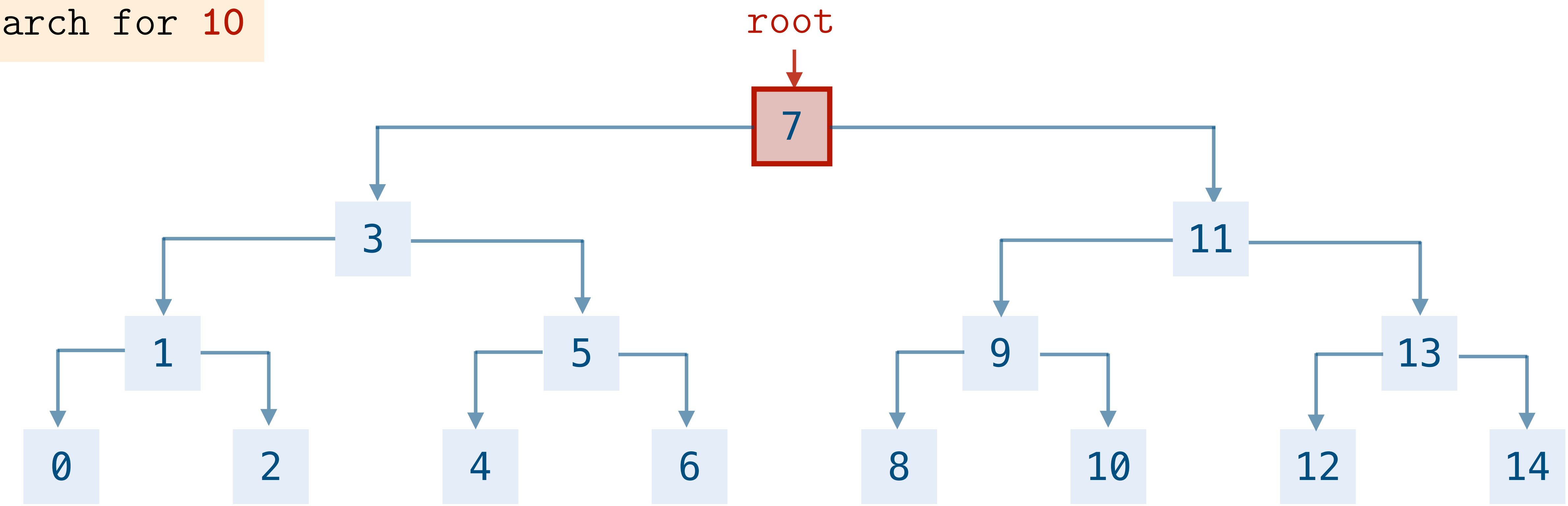
Binary Search Trees



Informally. A binary search tree (BST) is a binary tree that allows performing binary search!

Binary Search Trees

`Search for 10`



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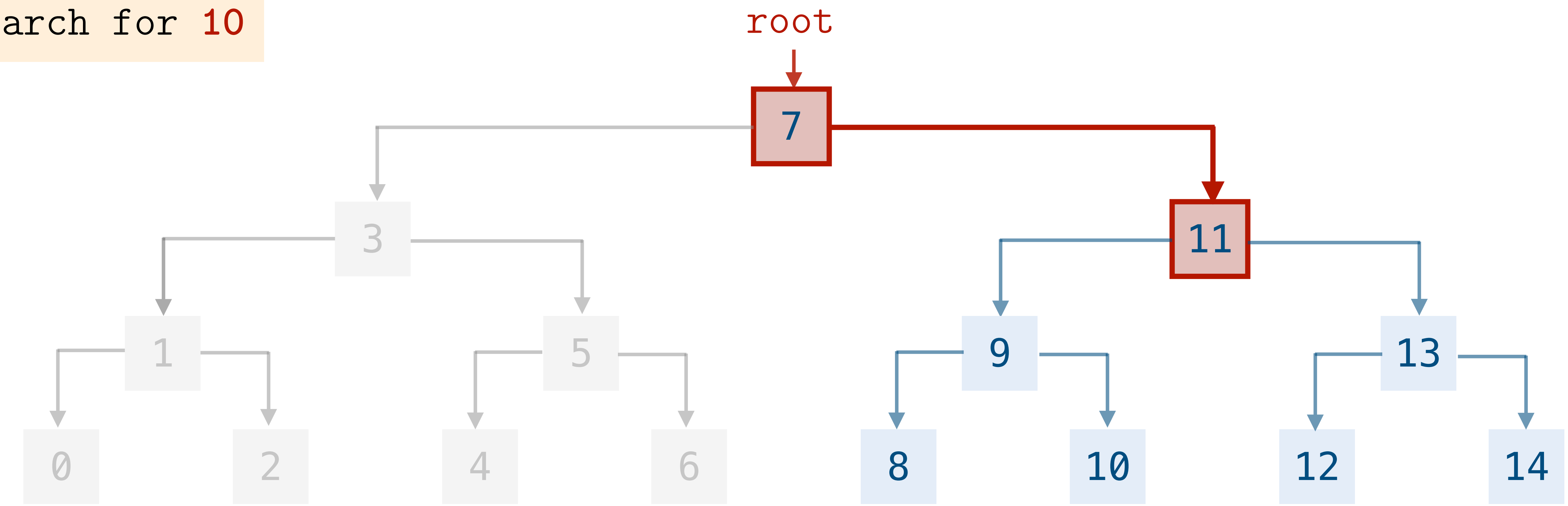

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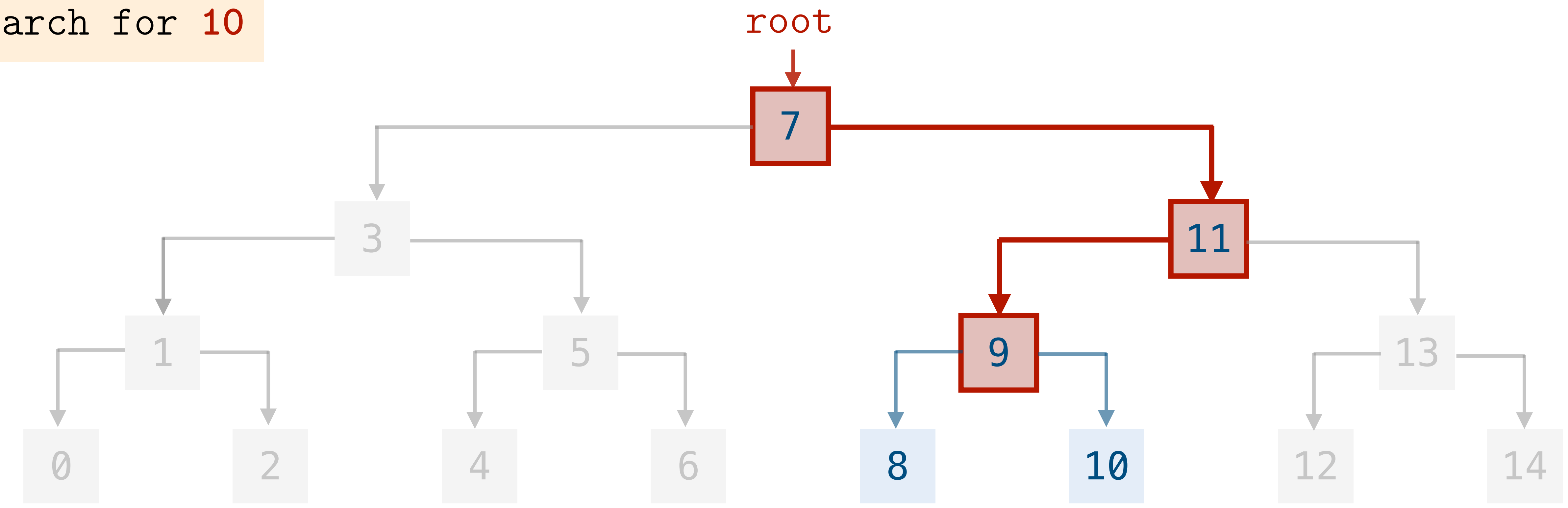

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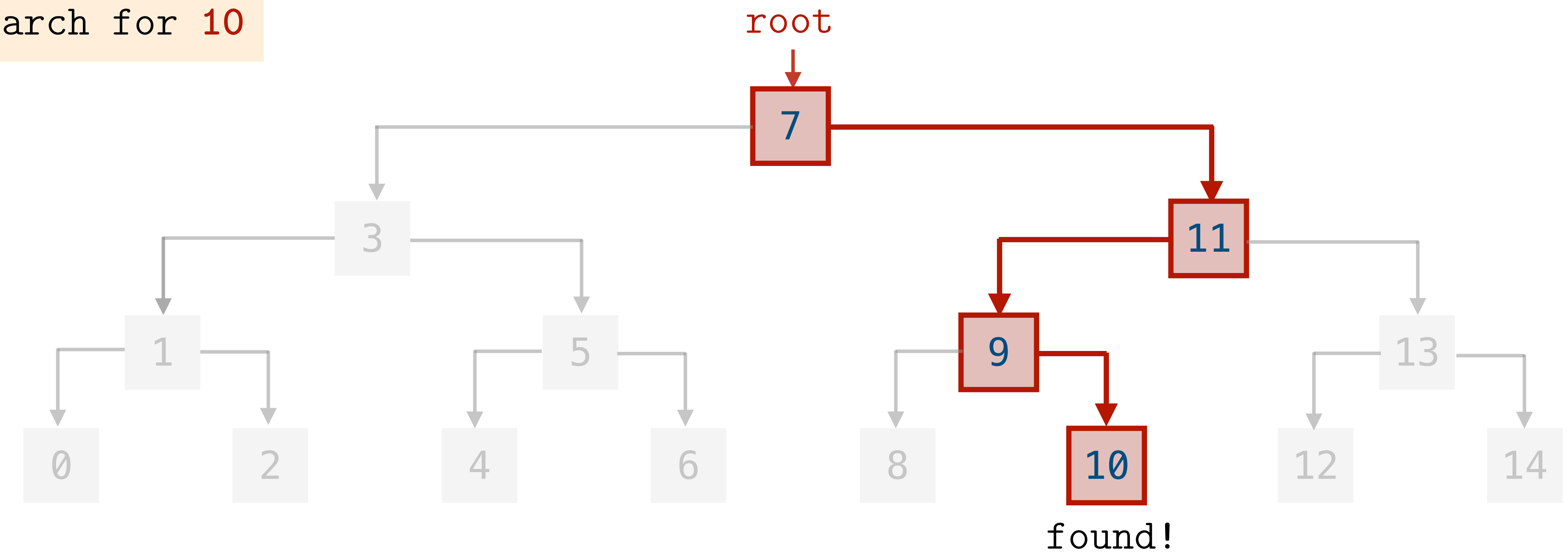
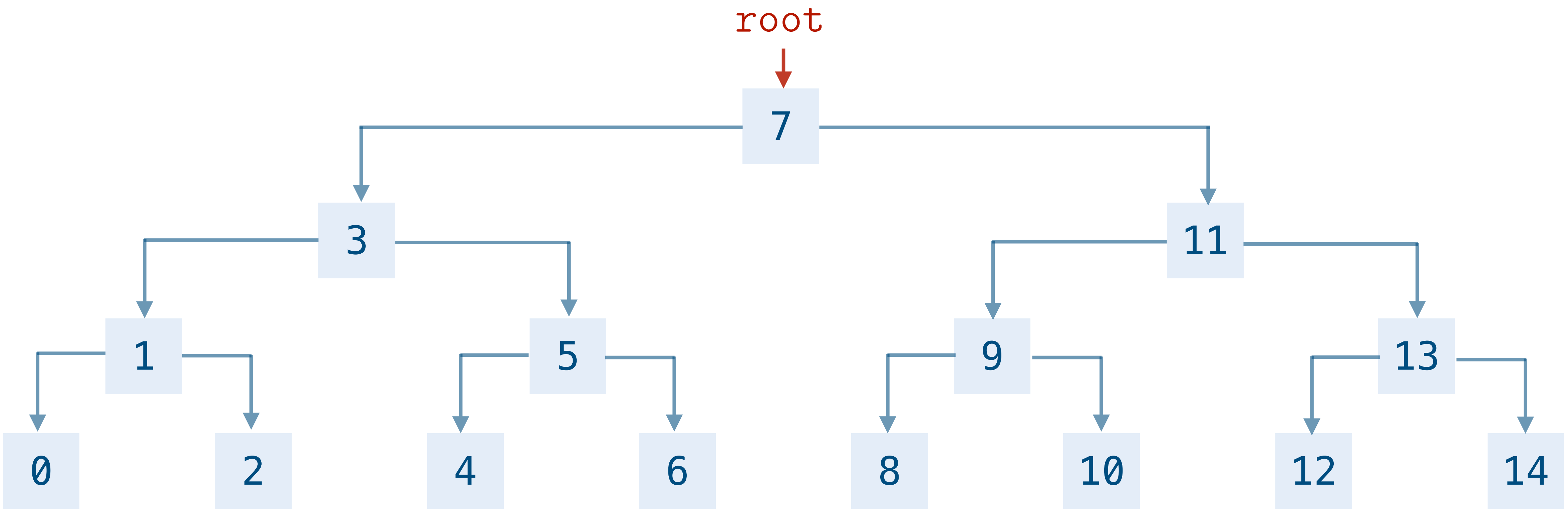


Why is binary search possible on such a tree?

Binary Search Trees



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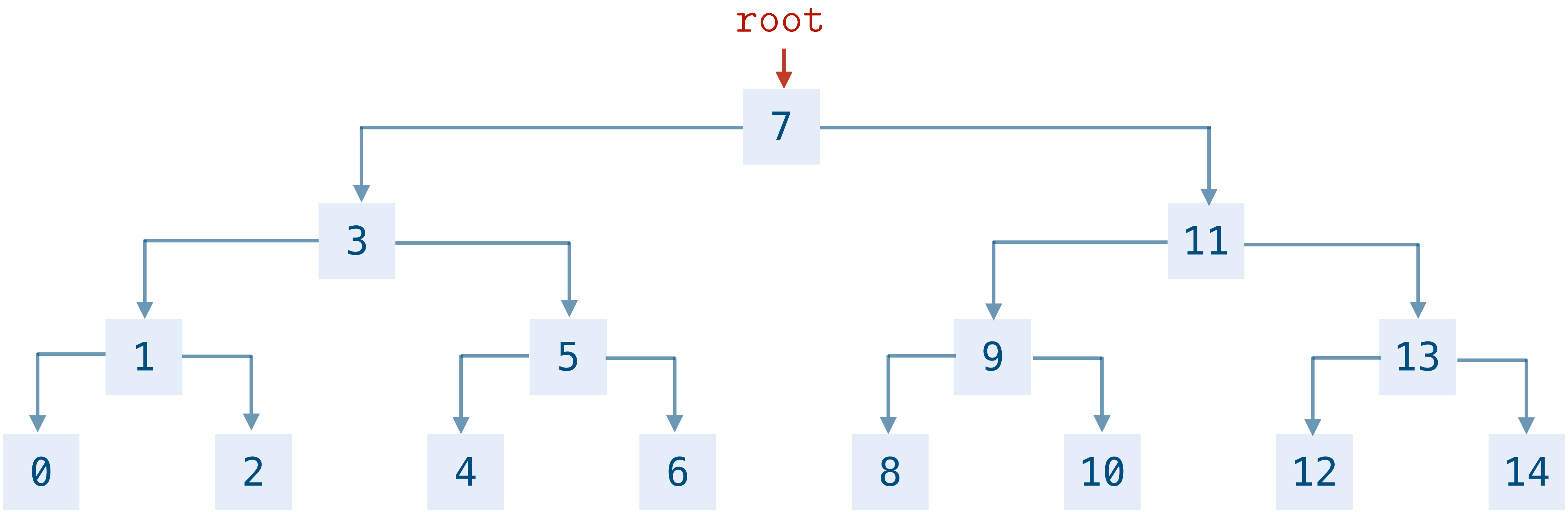
Binary Search Trees



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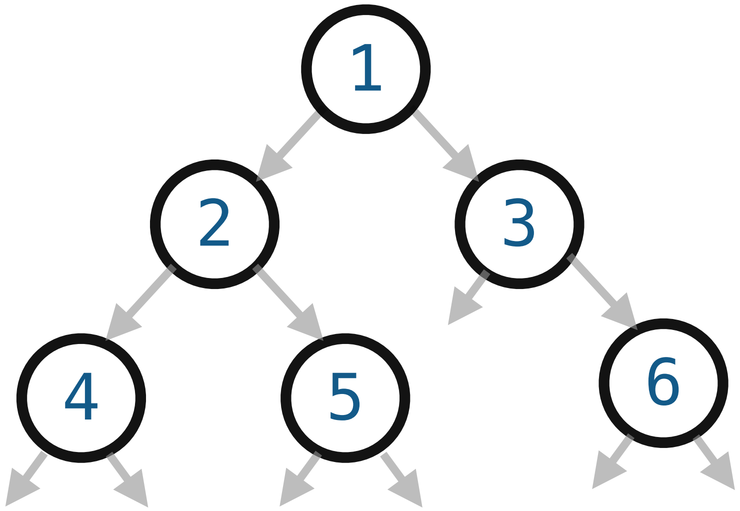

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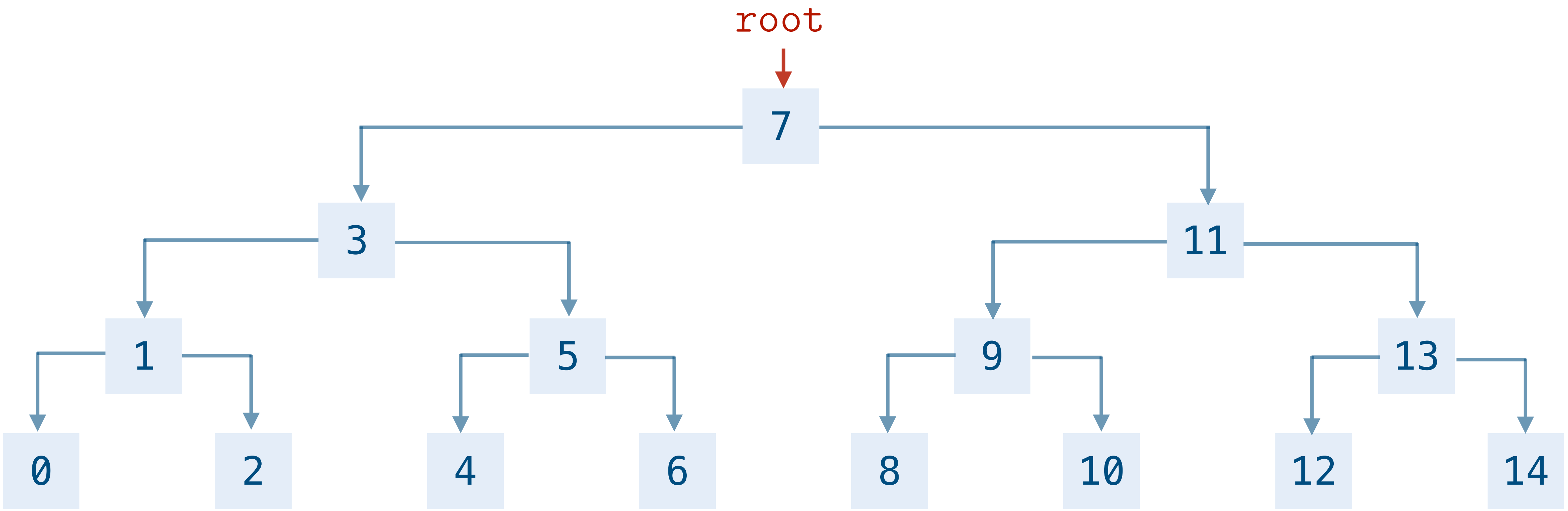

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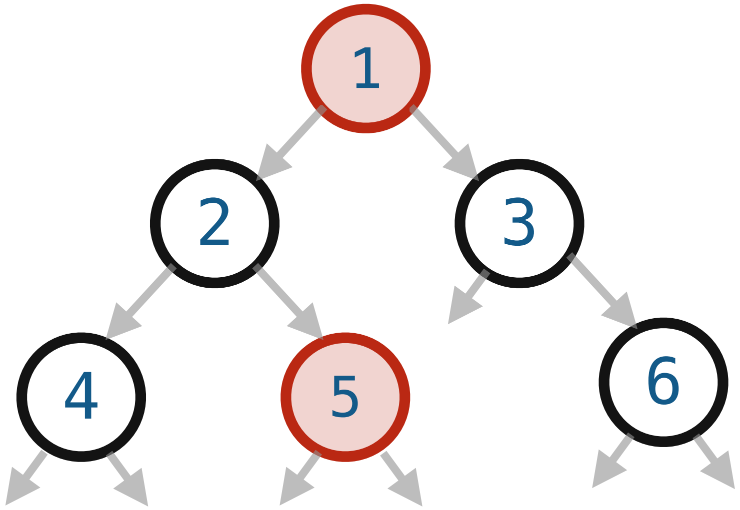
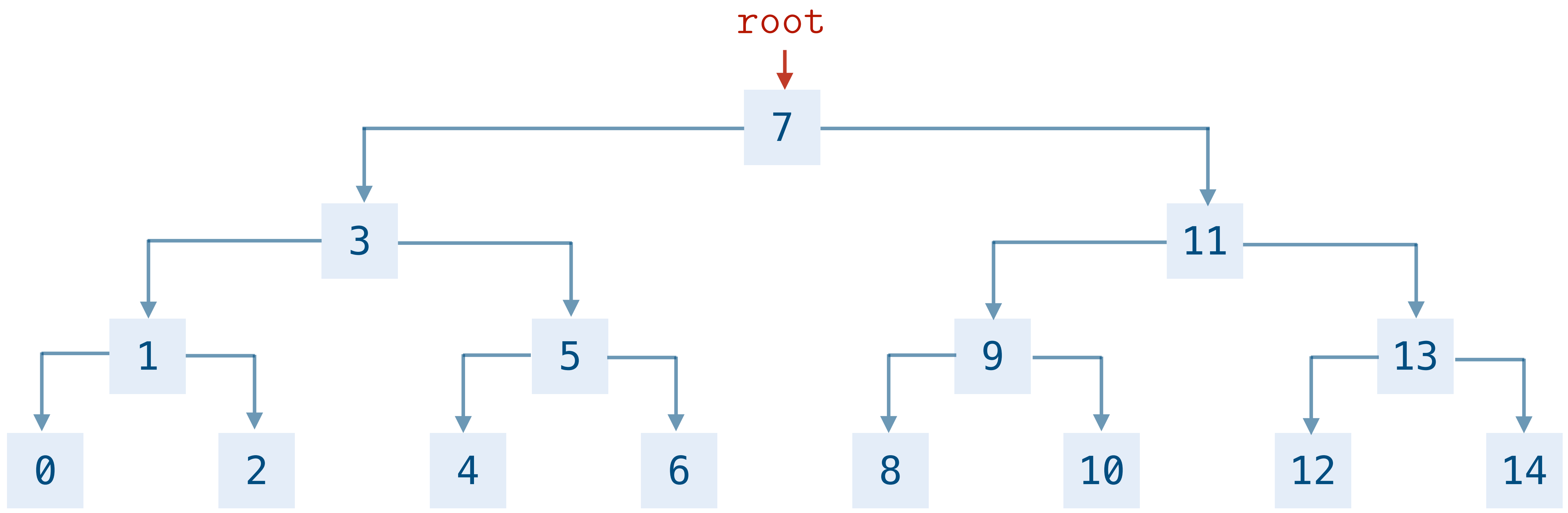
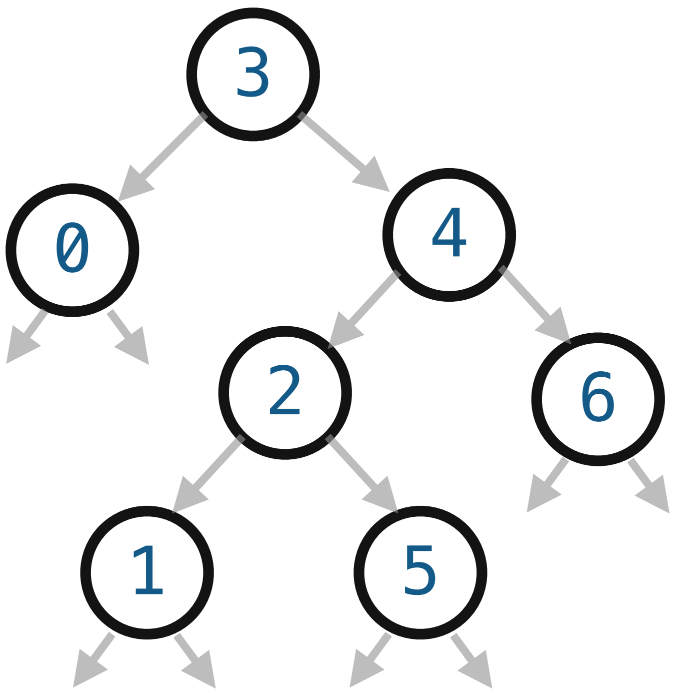


NOT a BST

5 is to the left of 1

Binary Search Trees





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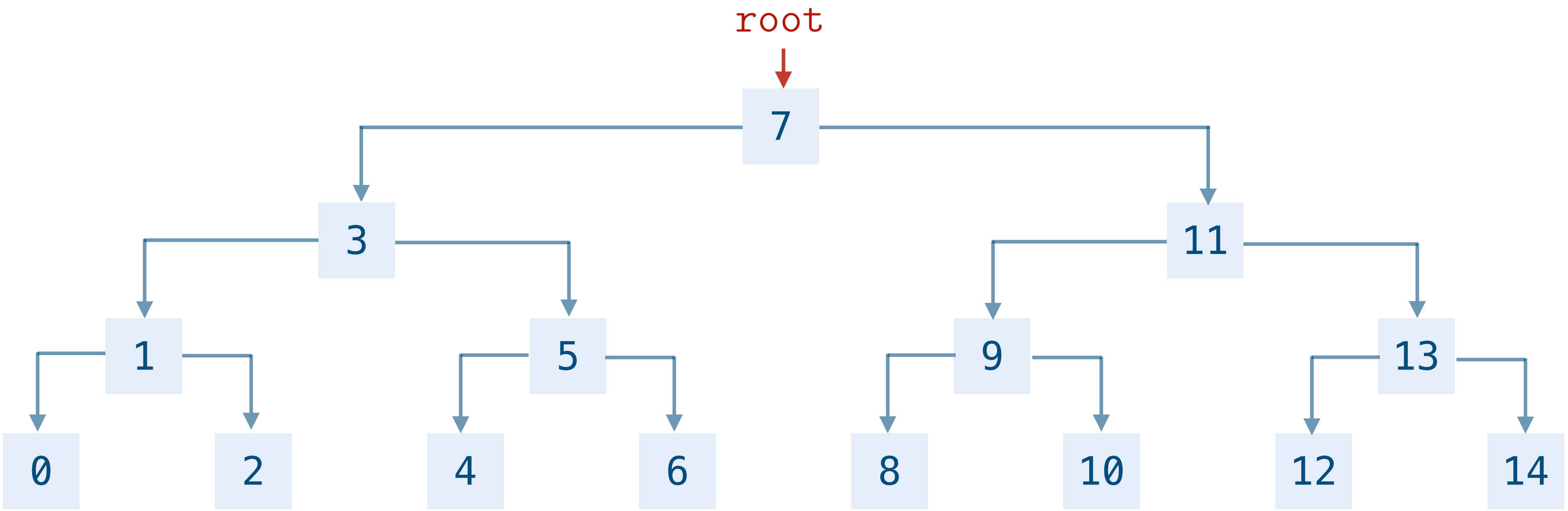

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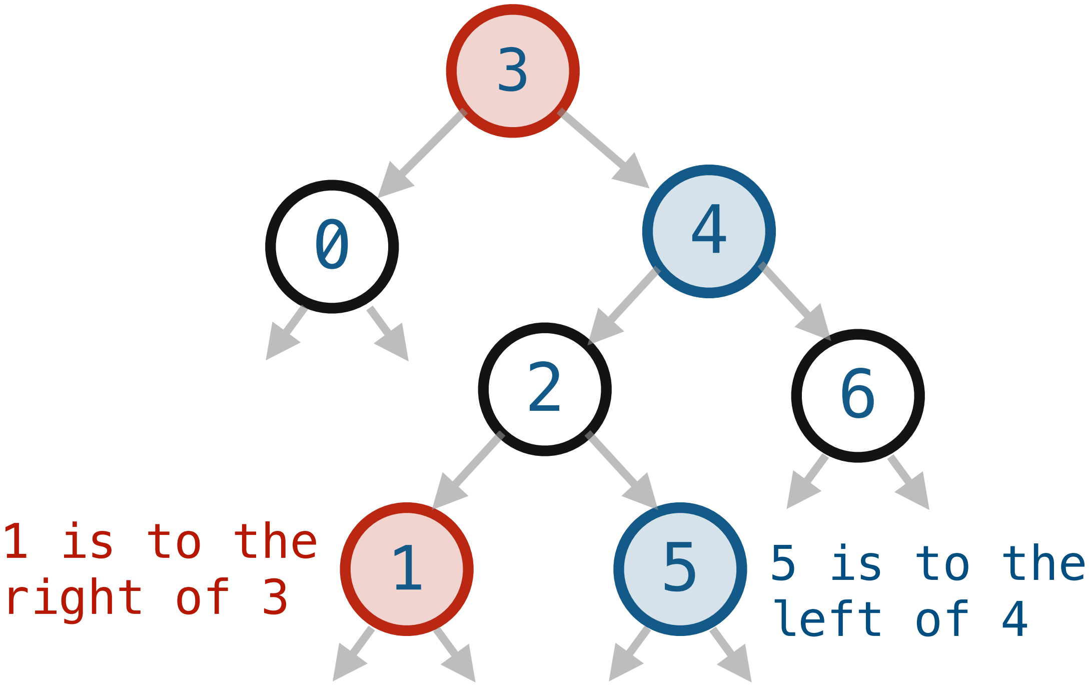
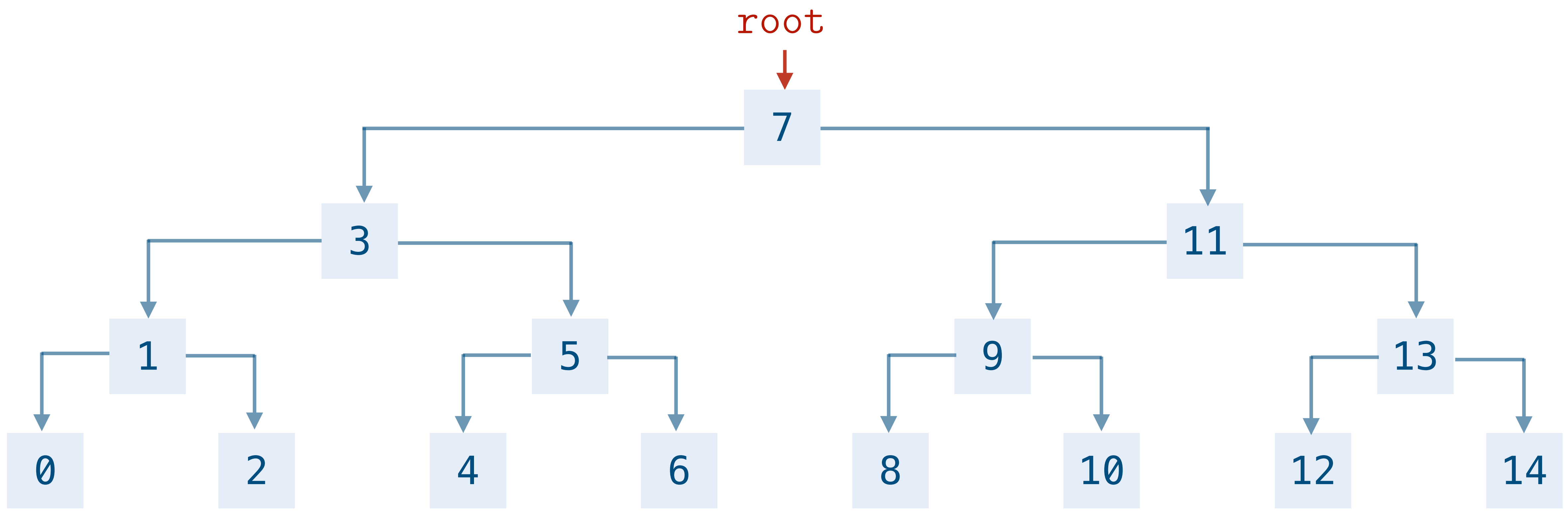


NOT a BST

Binary Search Trees



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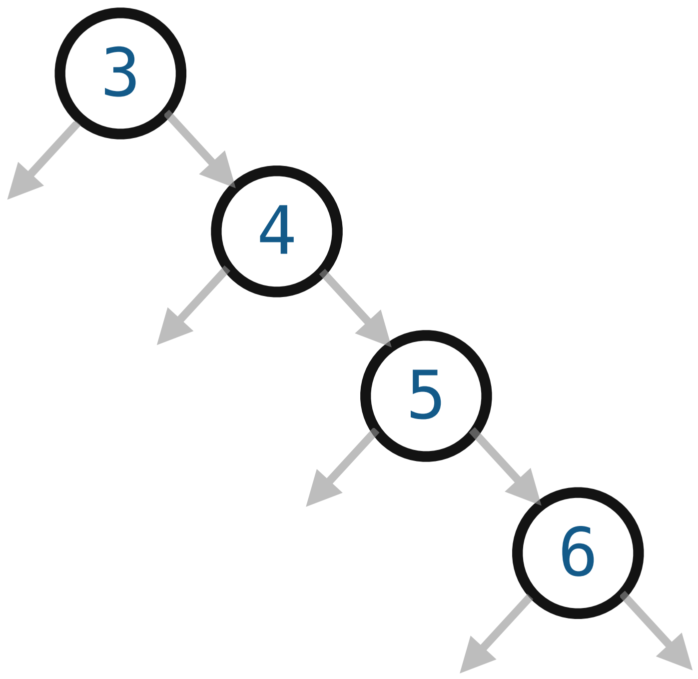

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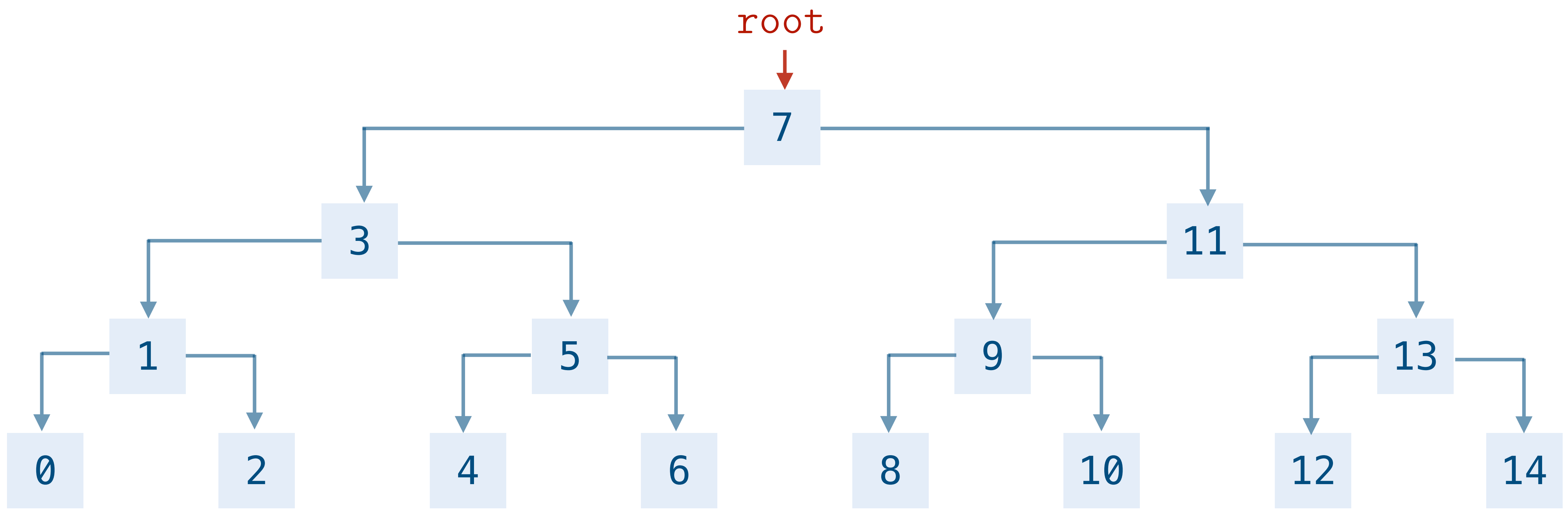

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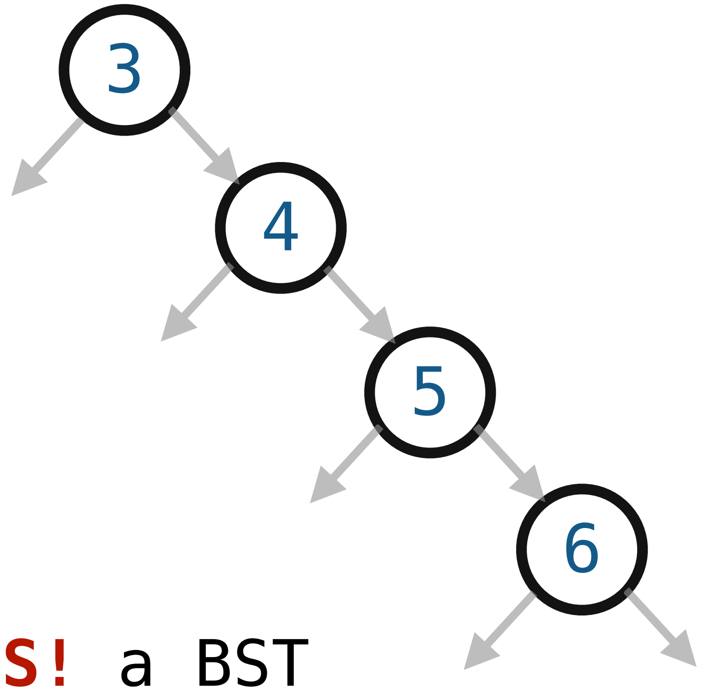
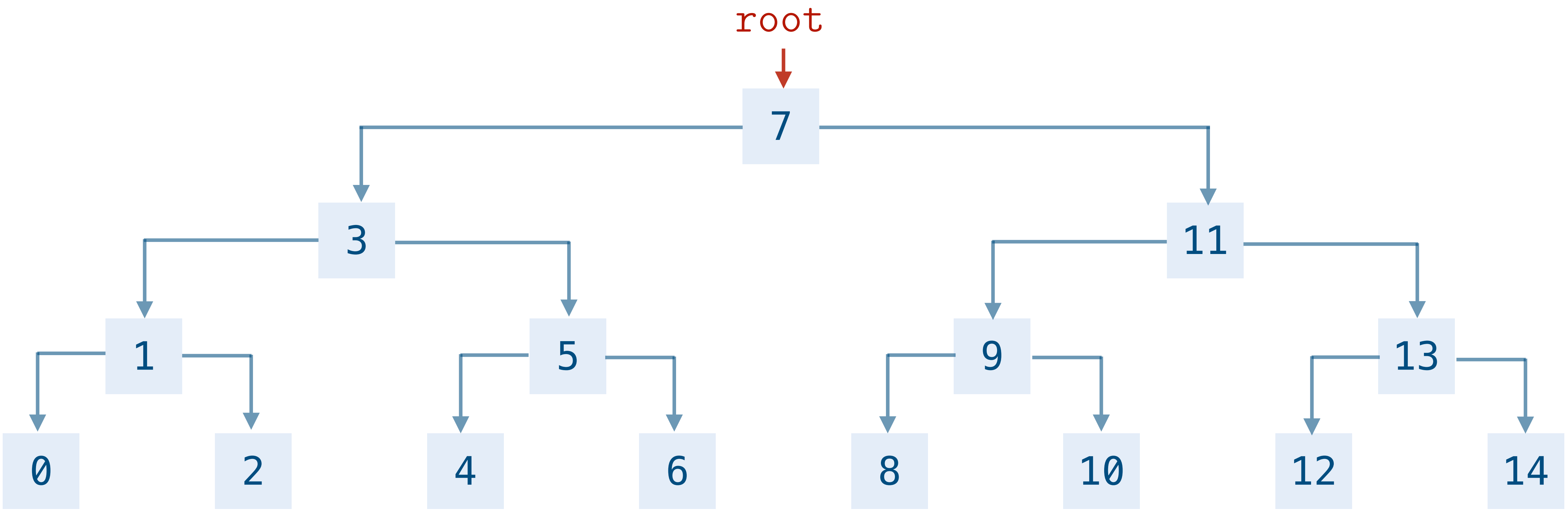


YES! a BST

Binary Search Trees



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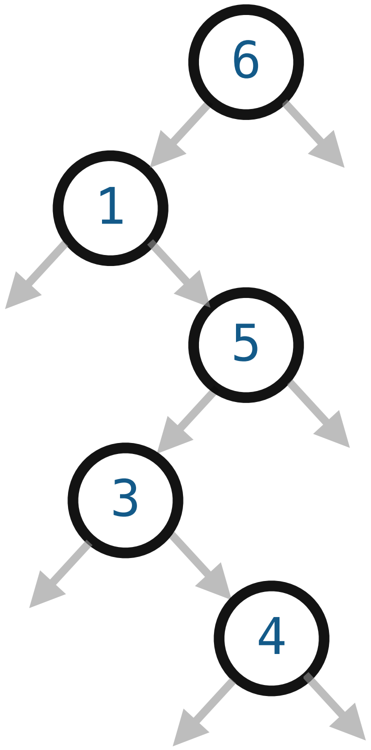

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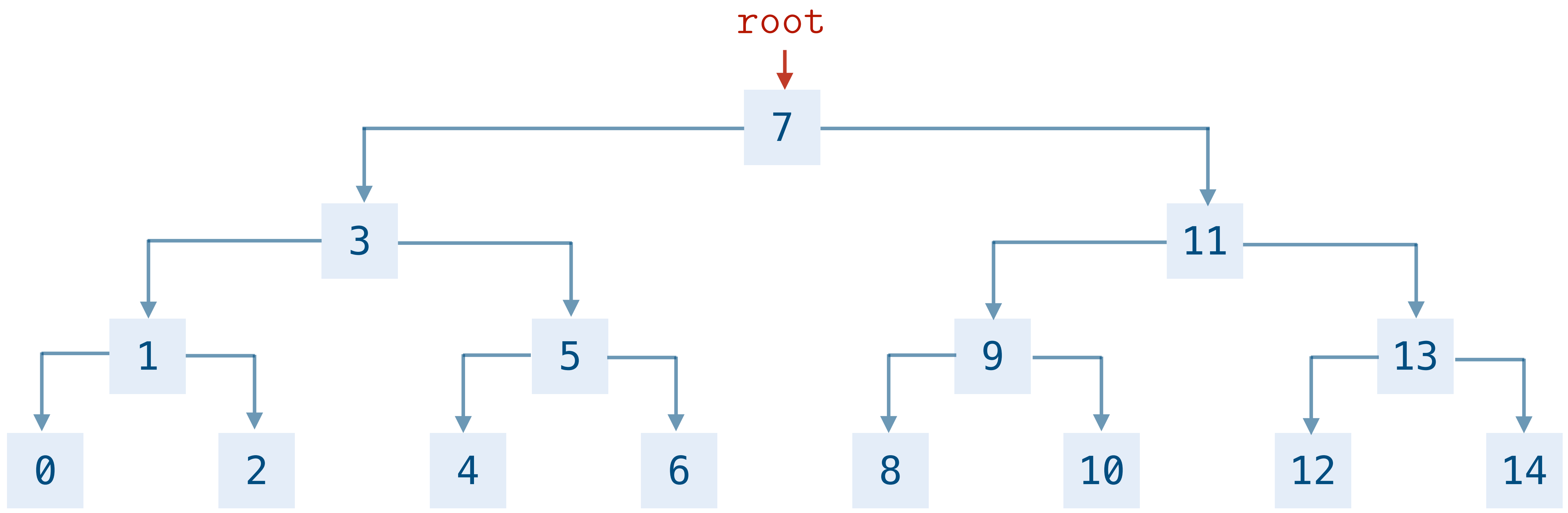

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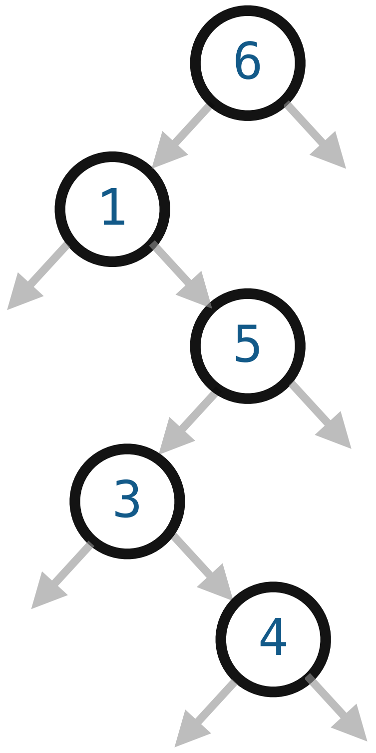

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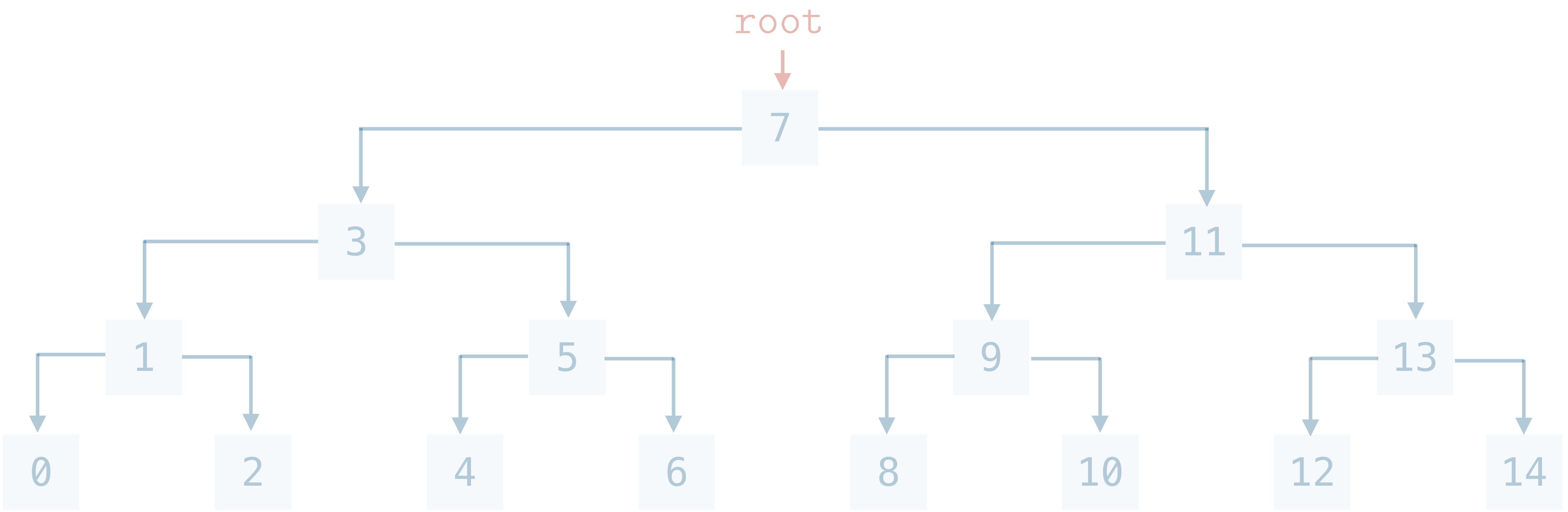
Binary Search Trees



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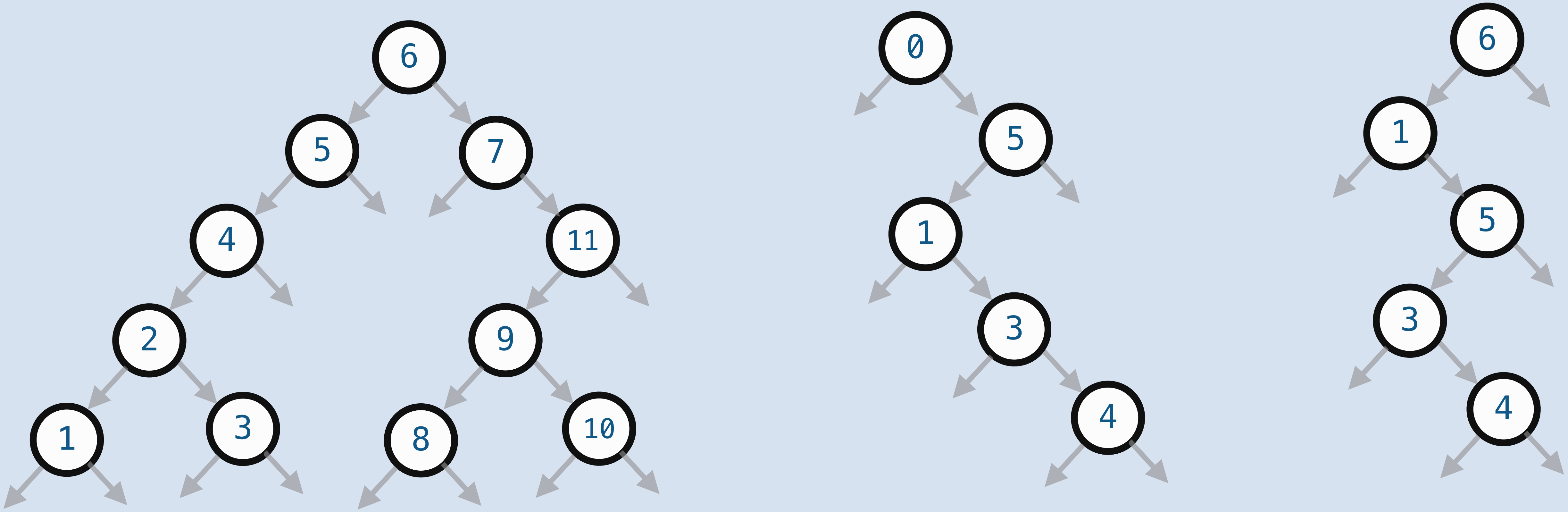
This applies also to every element in any sorted array!

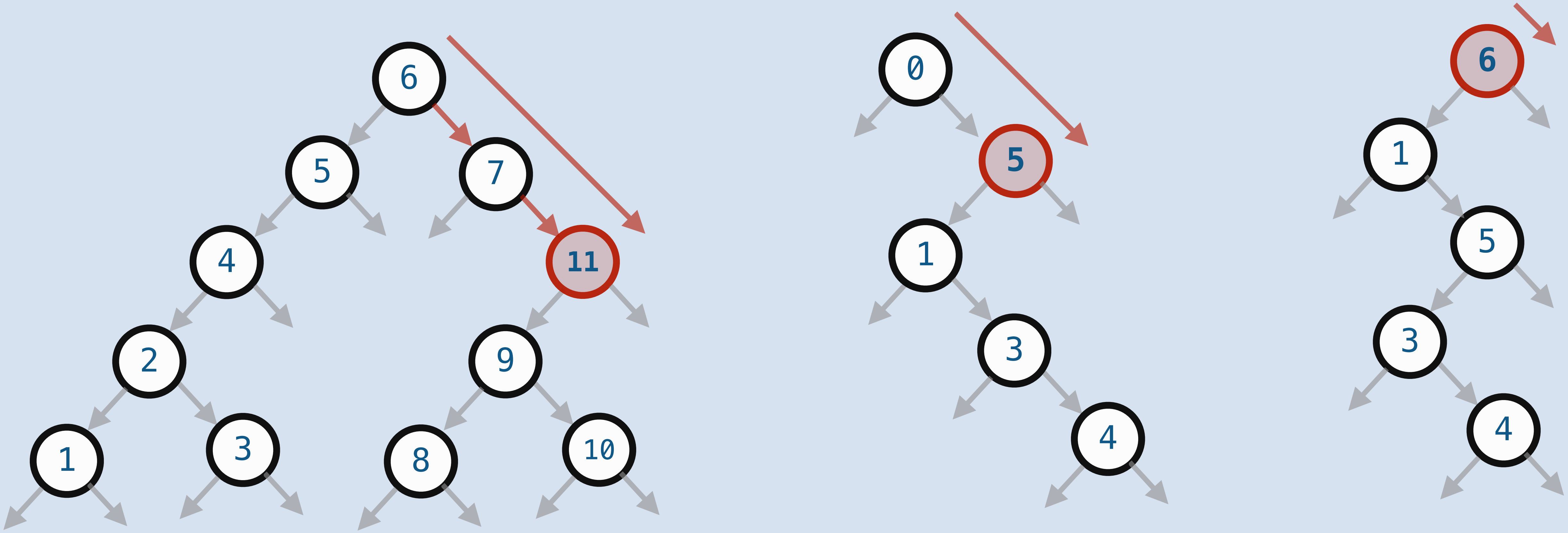
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14

Exercise



Question. Which node is always the *maximum* node in a BST?

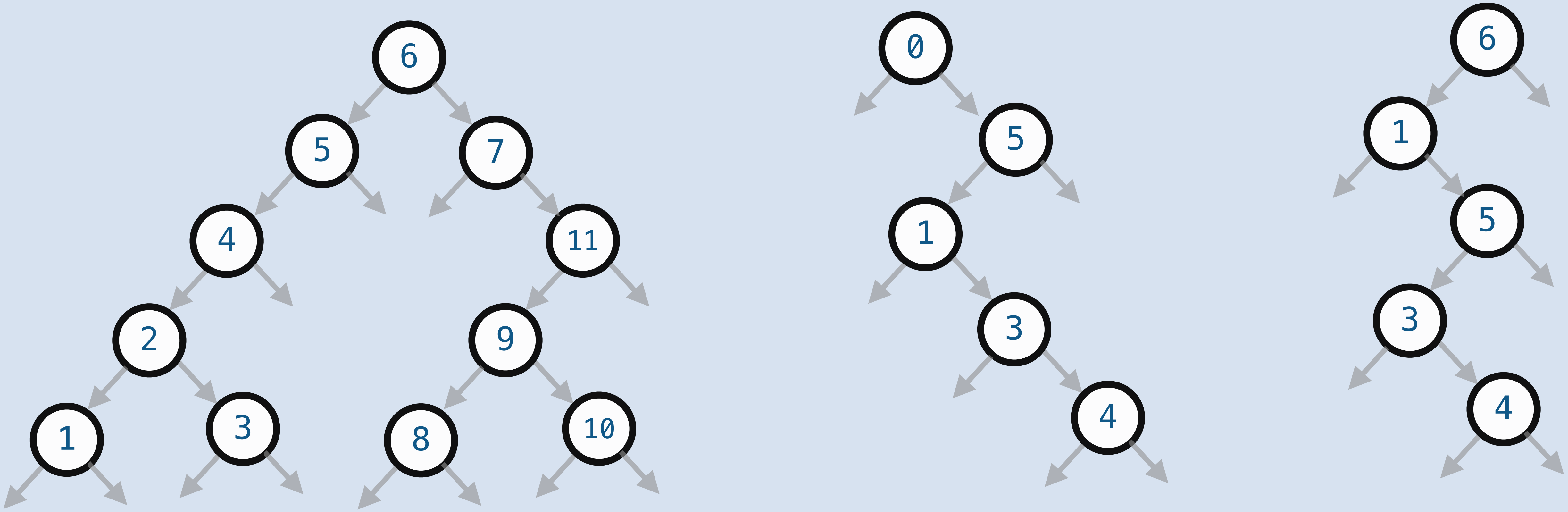
Exercise



Question. Which node is always the *maximum* node in a BST?

Answer. The right-most node.

Exercise

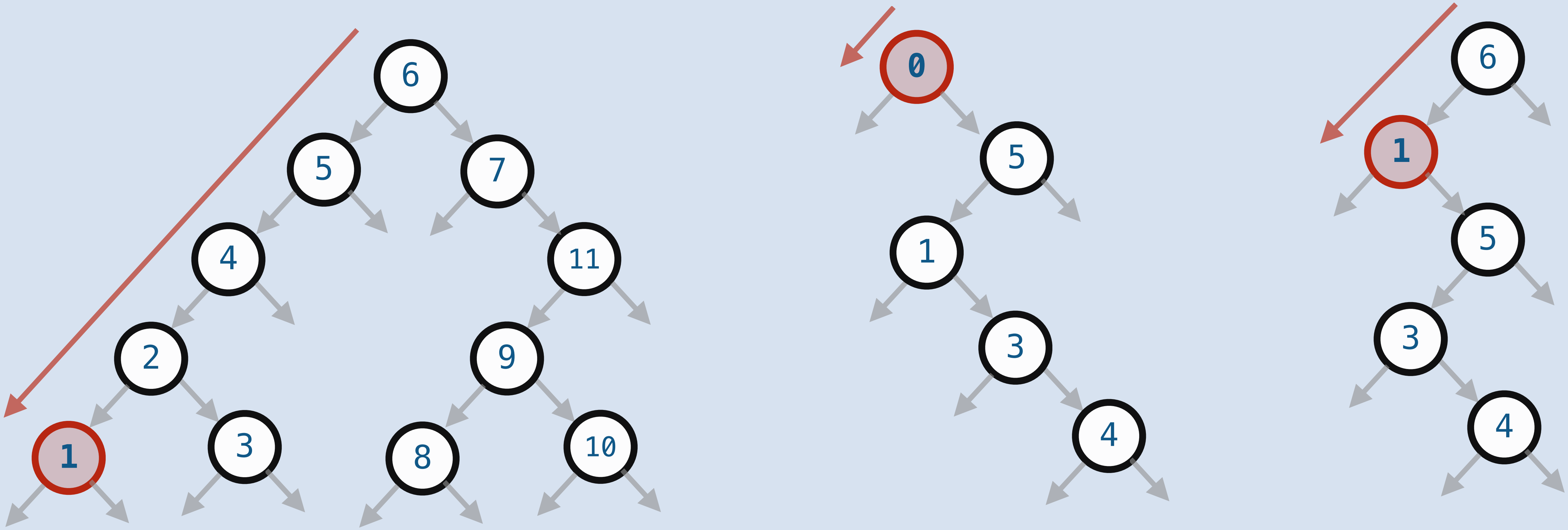


Question. Which node is always the *maximum* node in a BST?

Answer. The right-most node.

Question. Which node is always the *minimum* node in a BST?

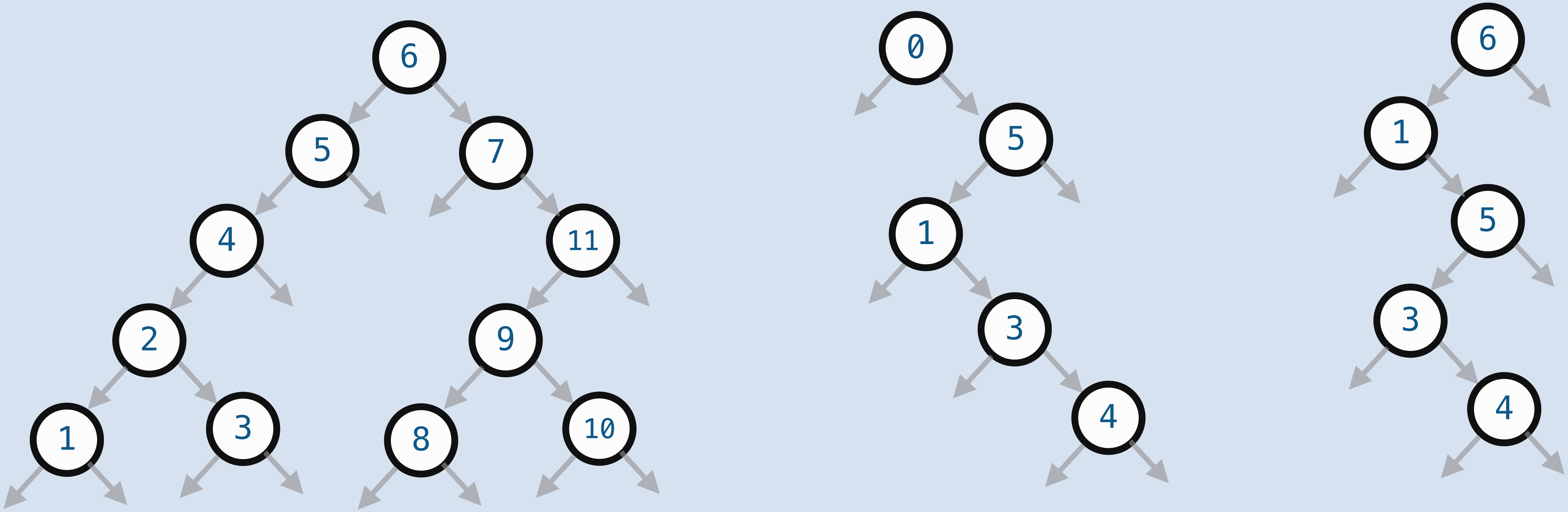
Exercise



Question. Which node is always the *maximum* node in a BST?
Answer. The right-most node.

Question. Which node is always the *minimum* node in a BST?
Answer. The left-most node.

Exercise



Question. Which node is always the *maximum* node in a BST?

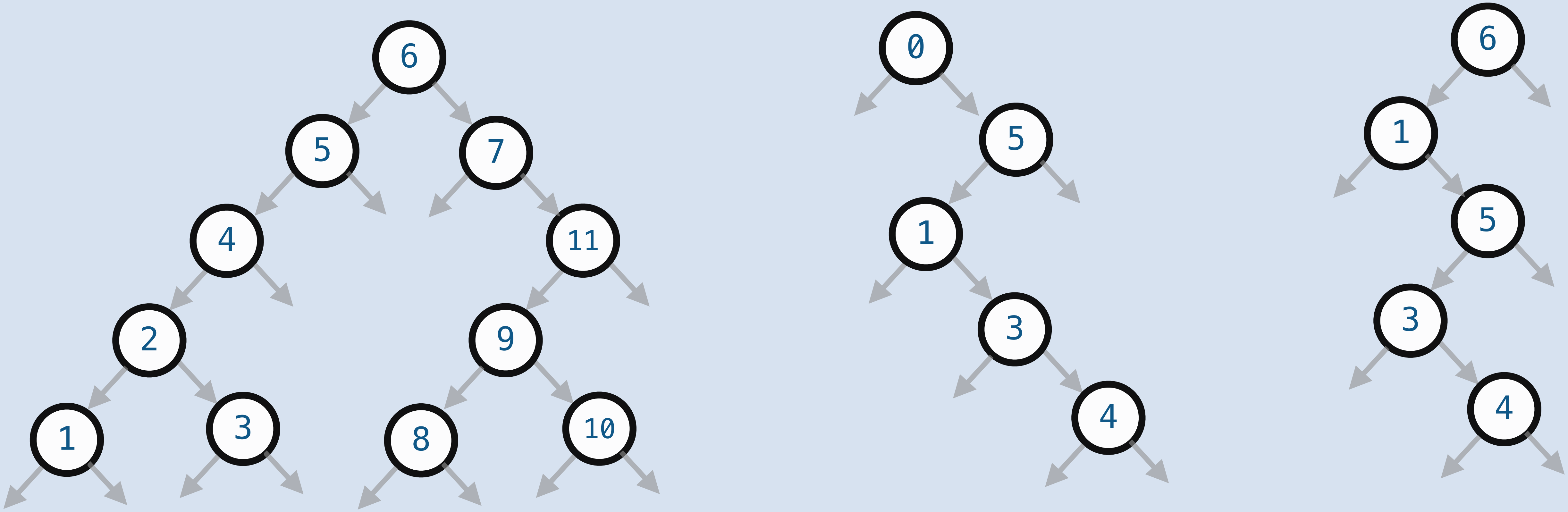
Answer. The right-most node.

Question. Which node is always the *minimum* node in a BST?

Answer. The left-most node.

Question. Which node is always the *median* node in a BST?

Exercise

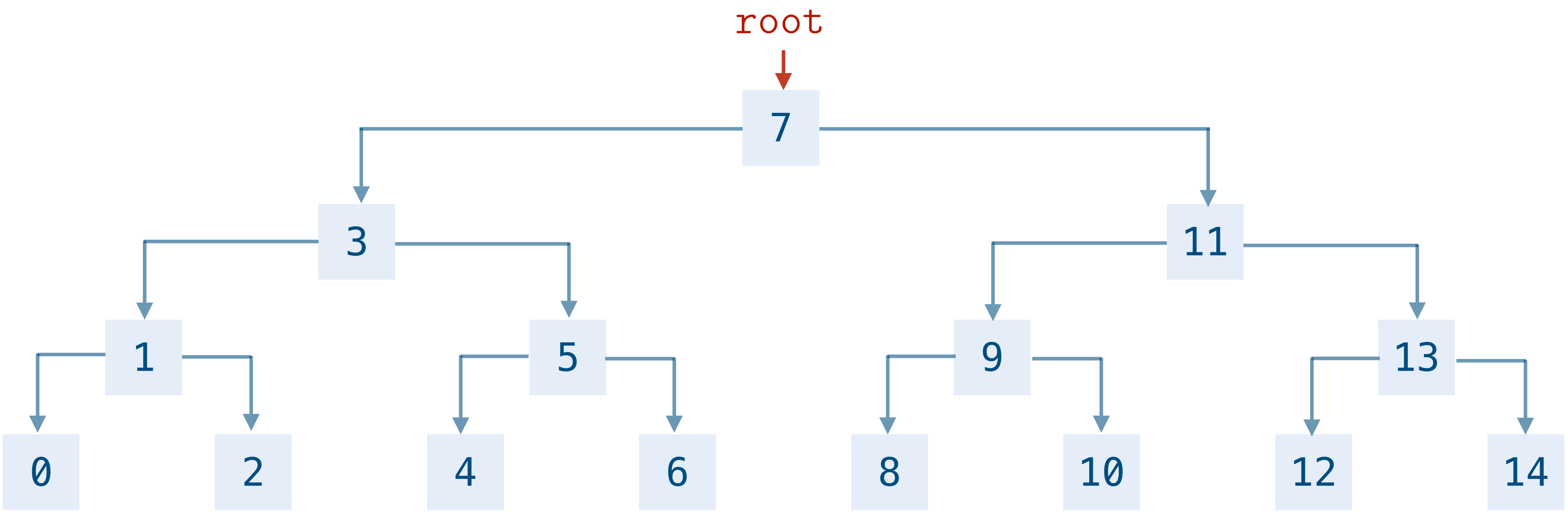


Question. Which node is always the *maximum* node in a BST?
Answer. The right-most node.

Question. Which node is always the *minimum* node in a BST?
Answer. The left-most node.

Question. Which node is always the *median* node in a BST?
Answer. Impossible to tell without performing a search for the median. *stay tuned!*

Binary Search Trees



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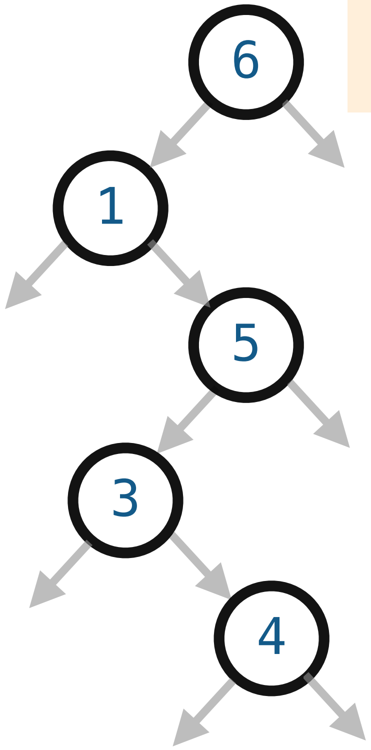
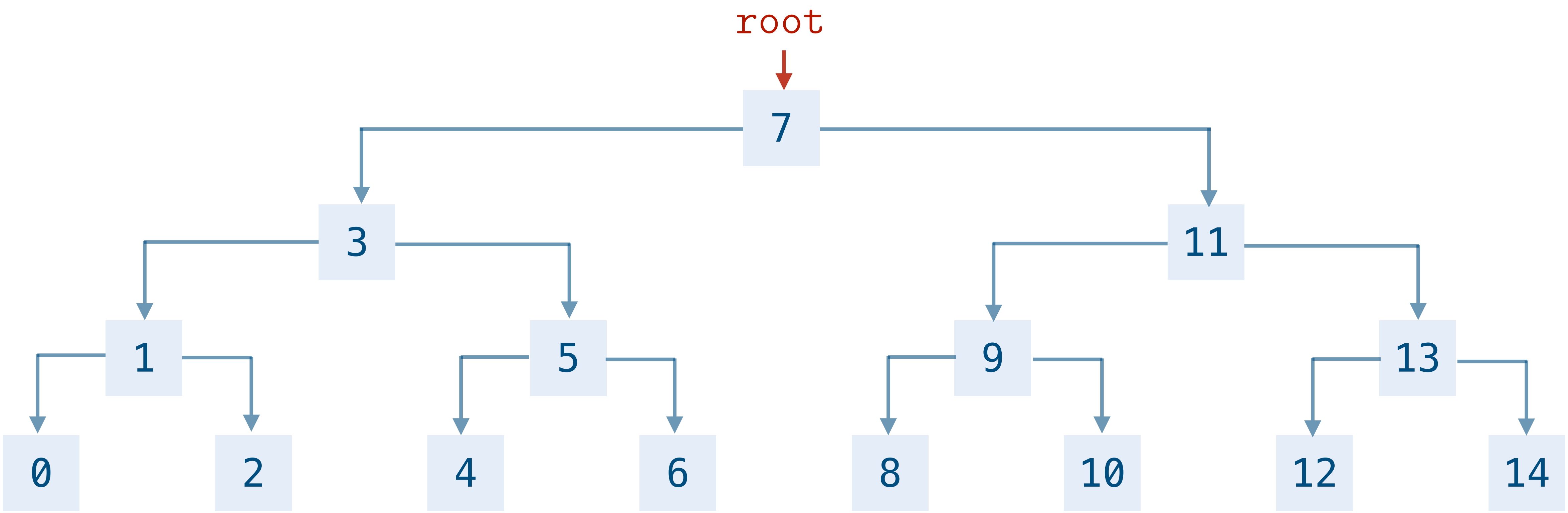


Search for 2

Binary Search Trees



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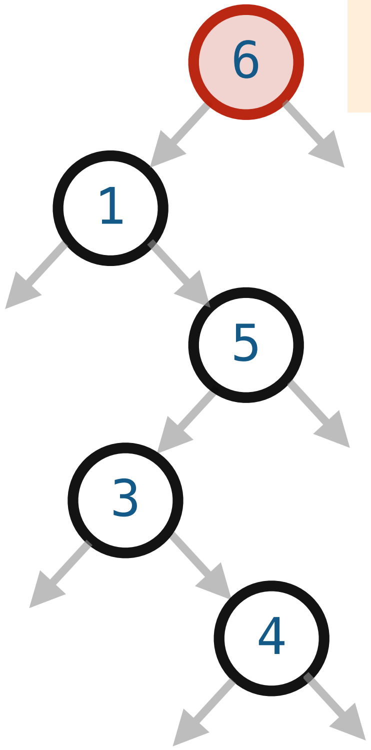
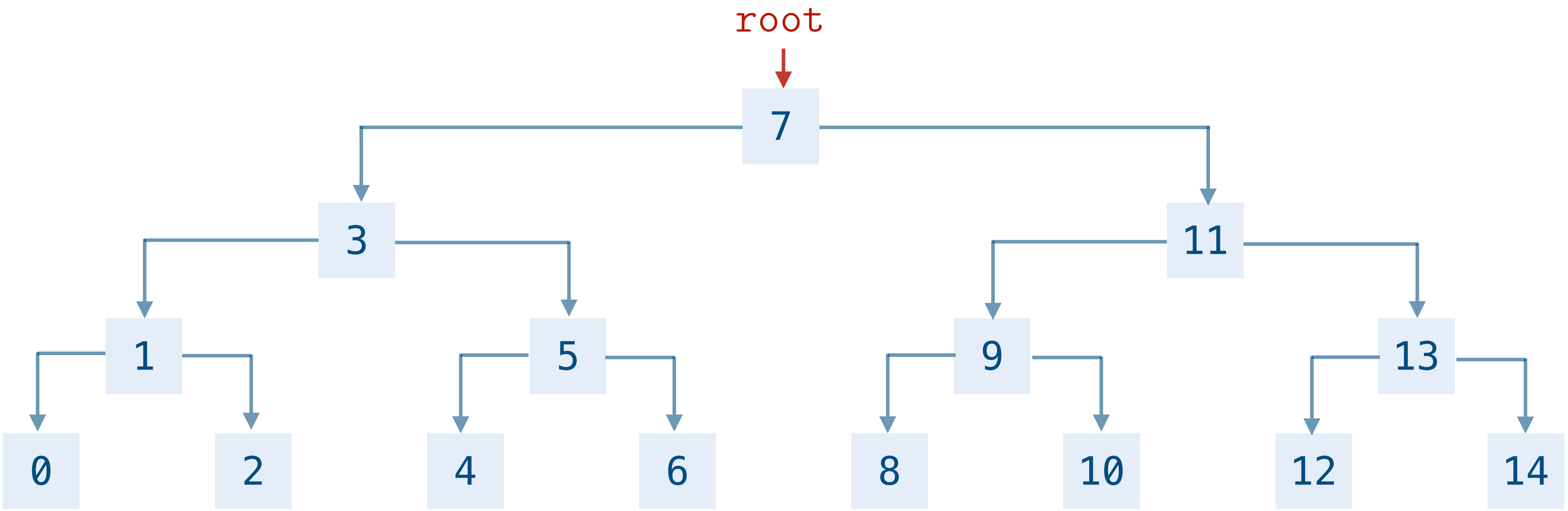


`Search for 2`

Binary Search Trees



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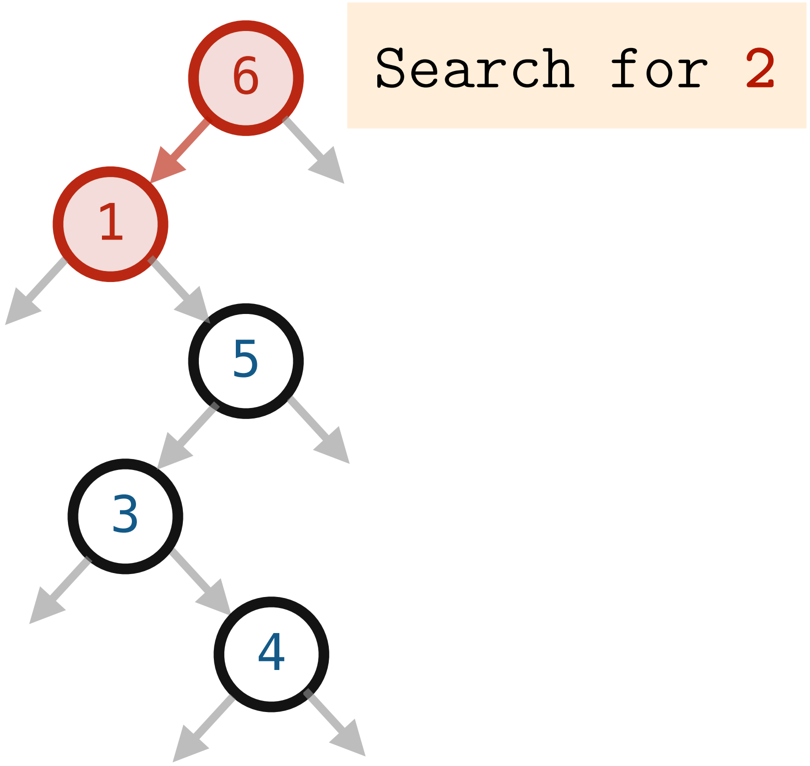

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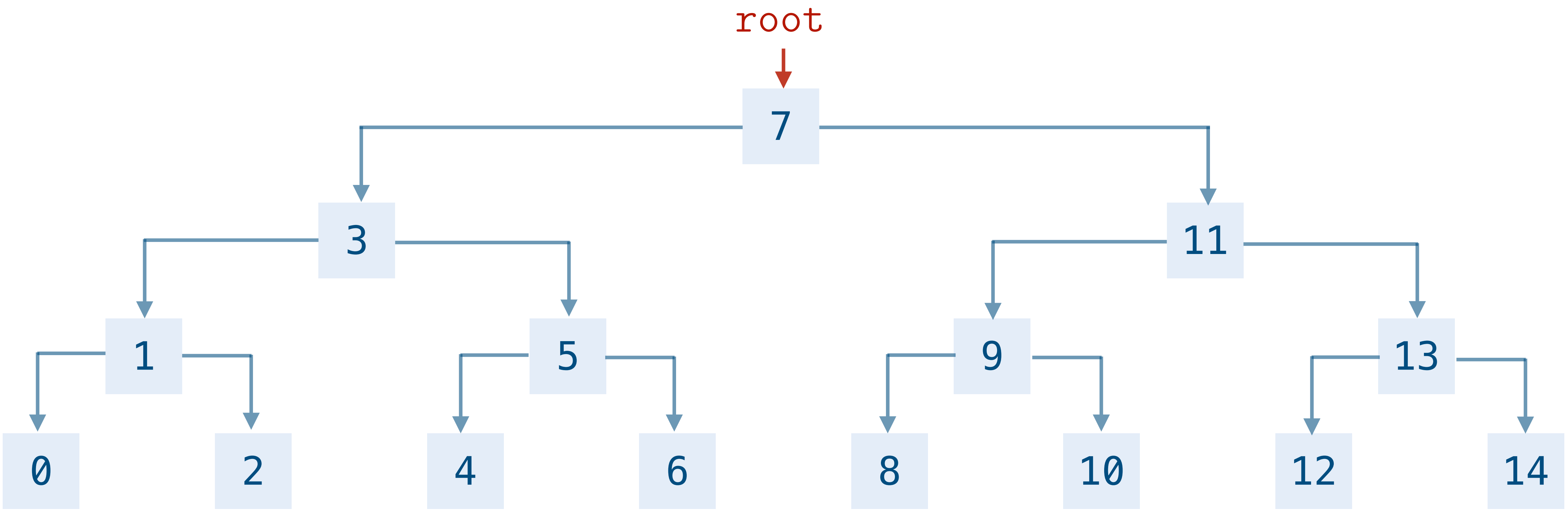

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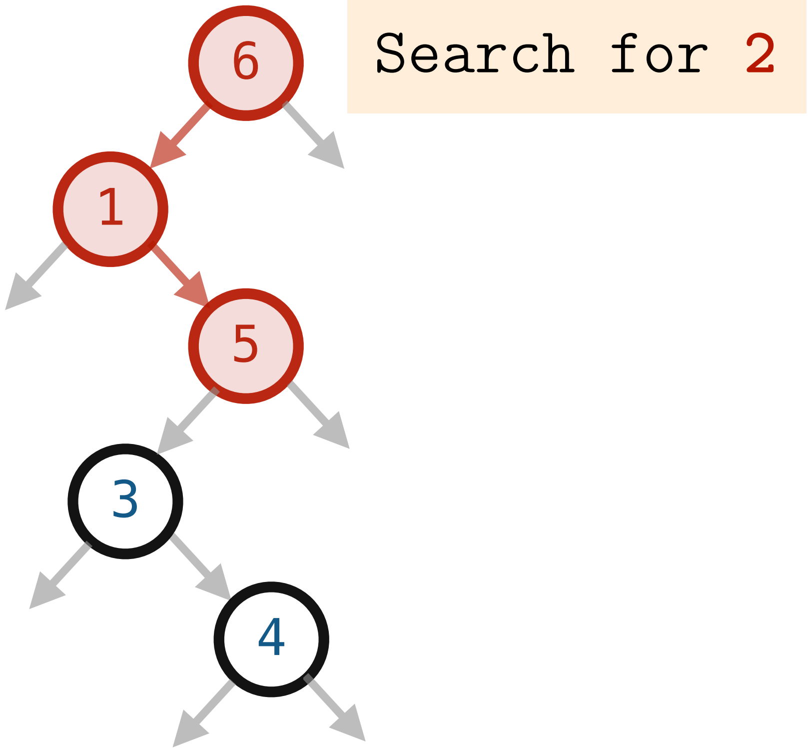

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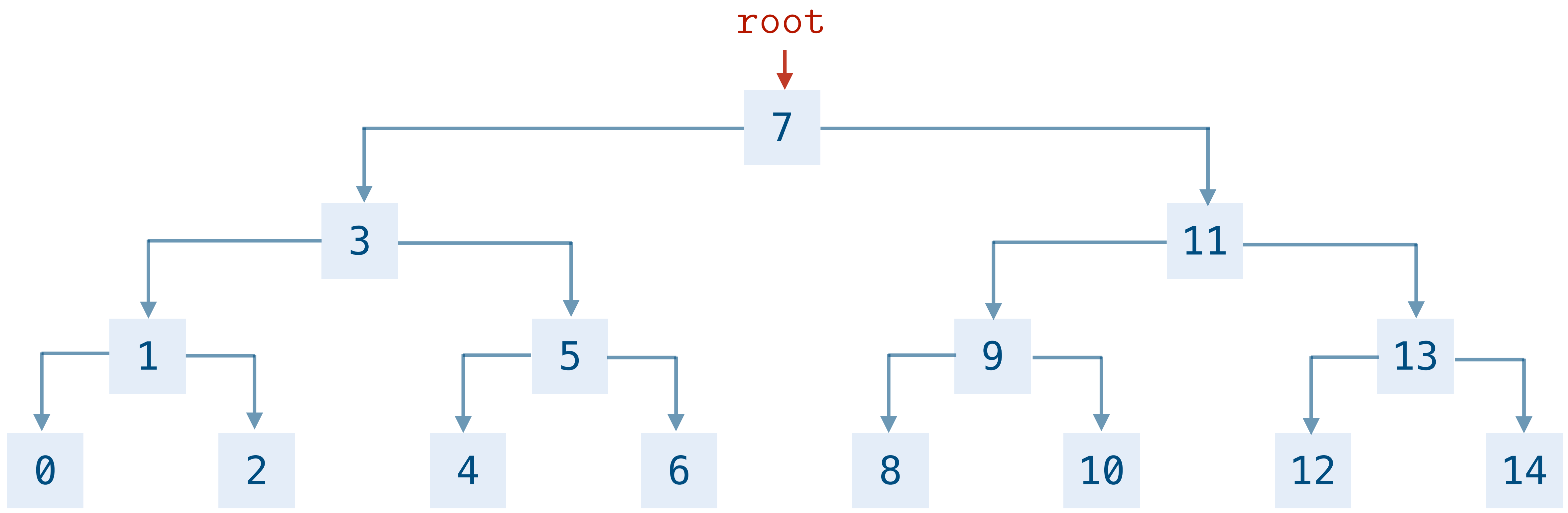

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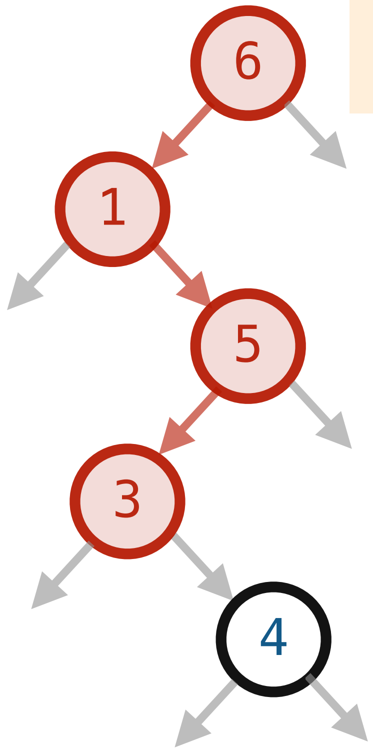

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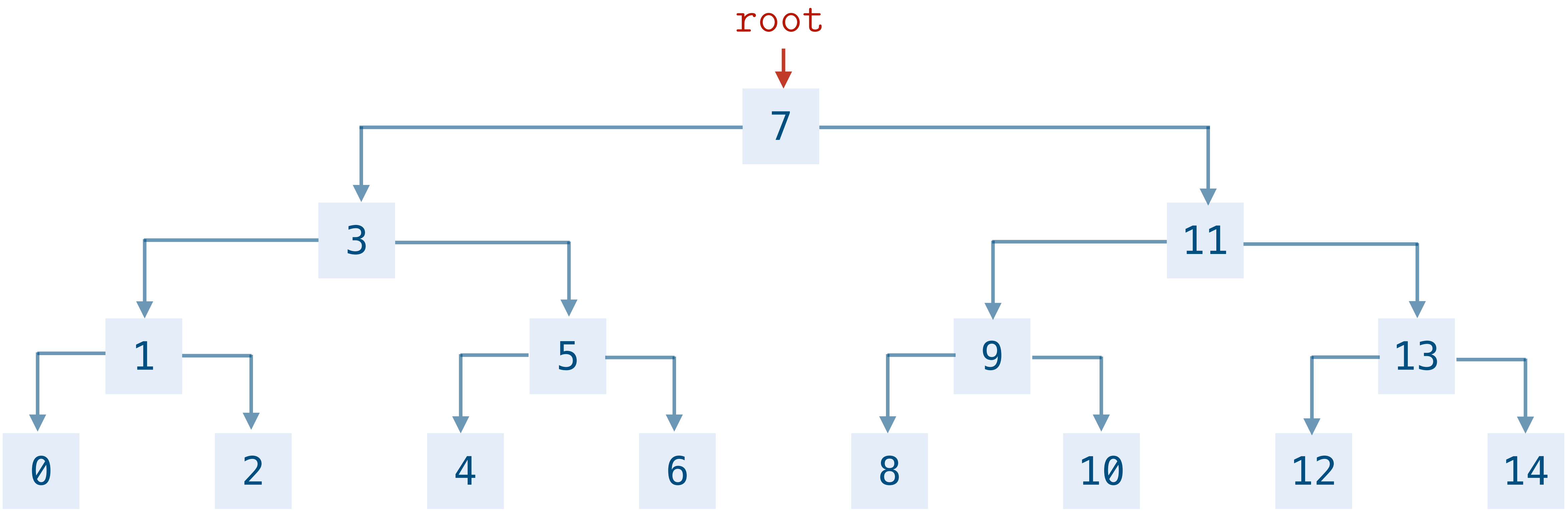
Binary Search Trees



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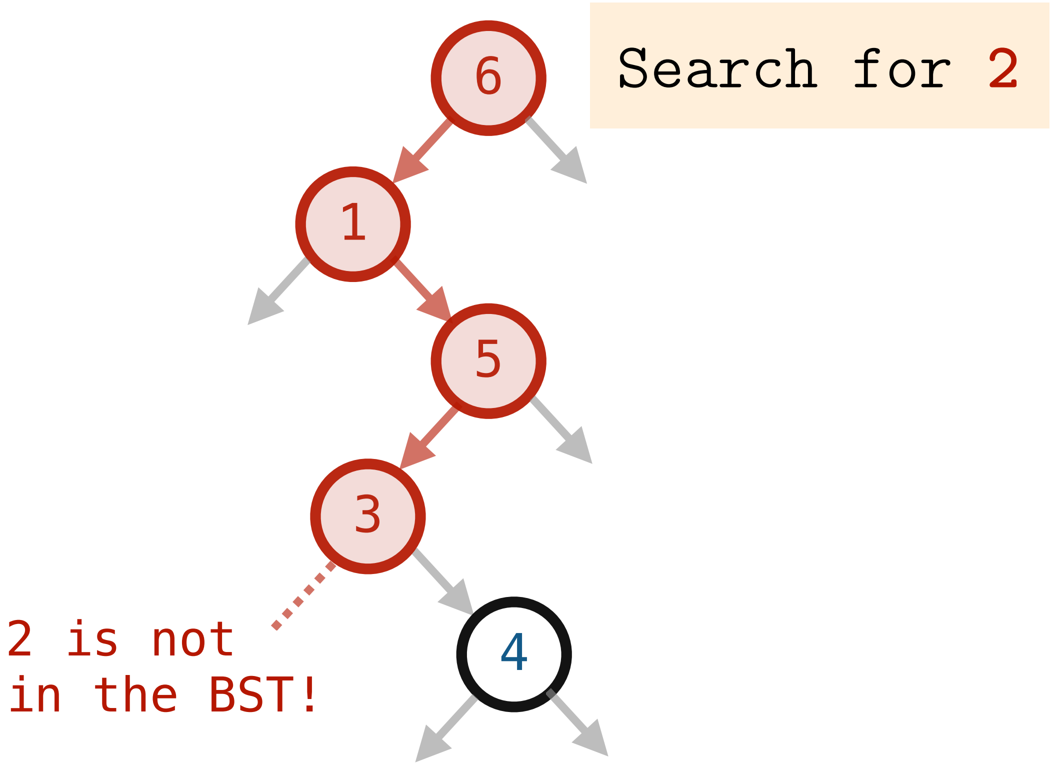

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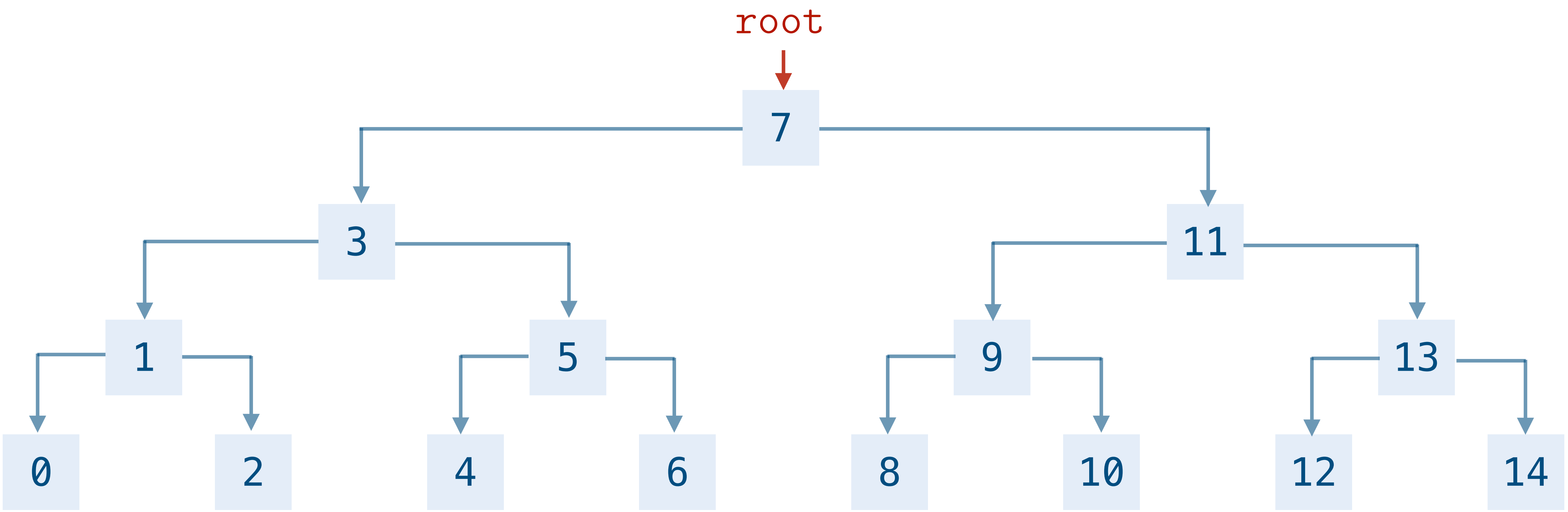

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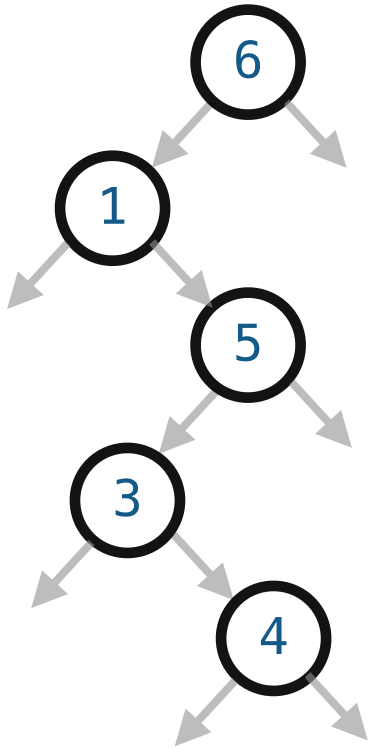
! Searching a BST requires *O*(*height*) compares.

what is the height of a BST?



YES! a BST

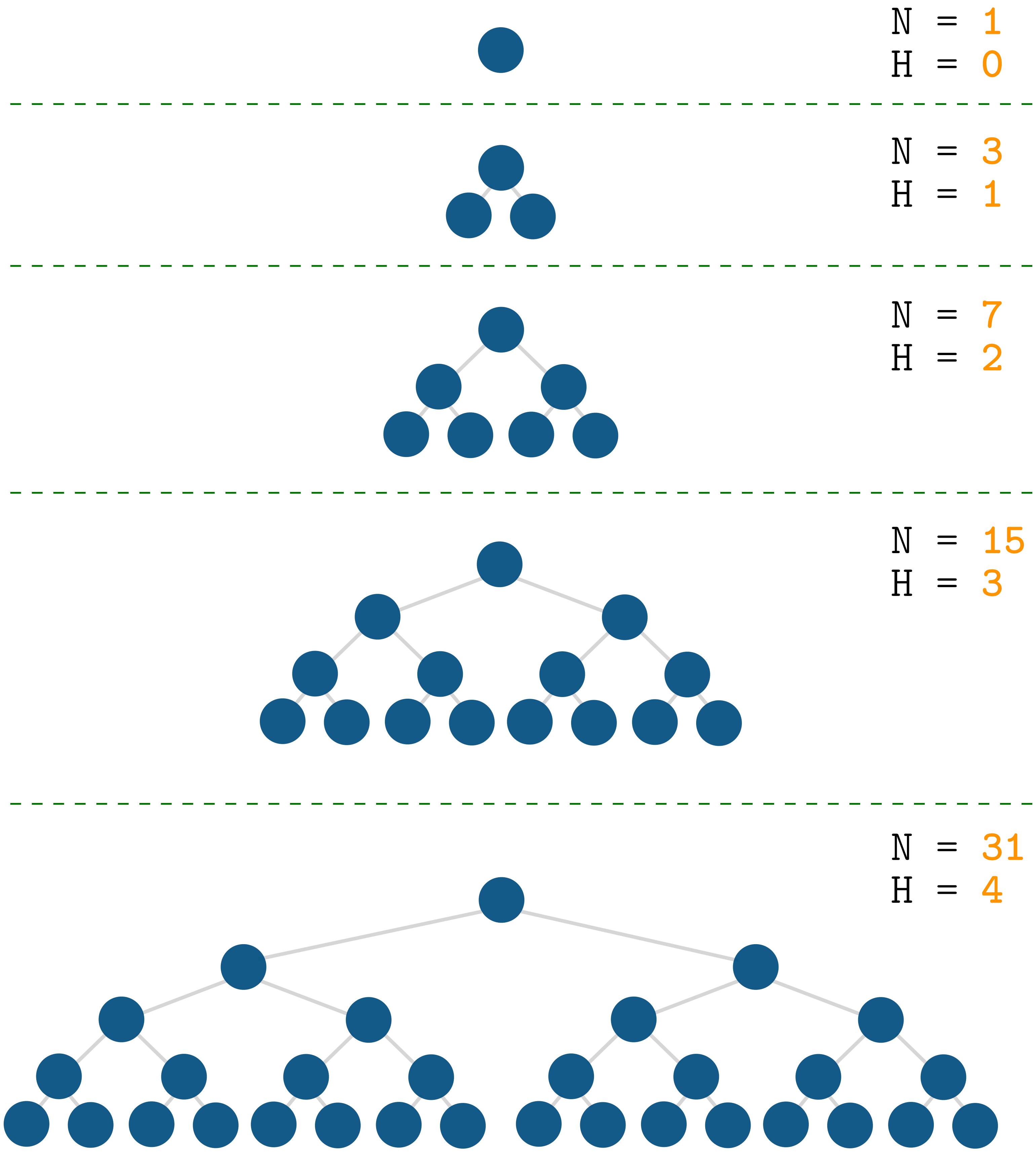
Perfect Binary Trees

Definition.

- All levels are full.
- All leafs are in the last level and all internal nodes have exactly two children.

CAUTION

You might find different definitions online. Some texts call this tree *complete*



Perfect Binary Trees

Definition.

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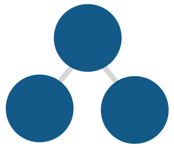
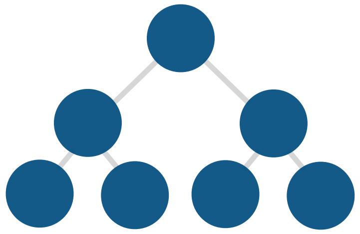
$$\text{\# of nodes } (N) = 2^{\text{height} + 1} - 1$$

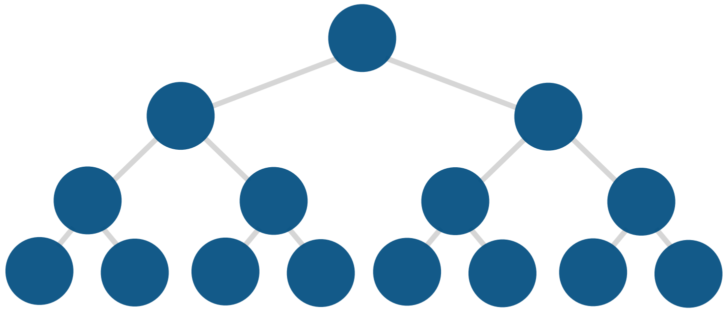
N = 1
H = 0



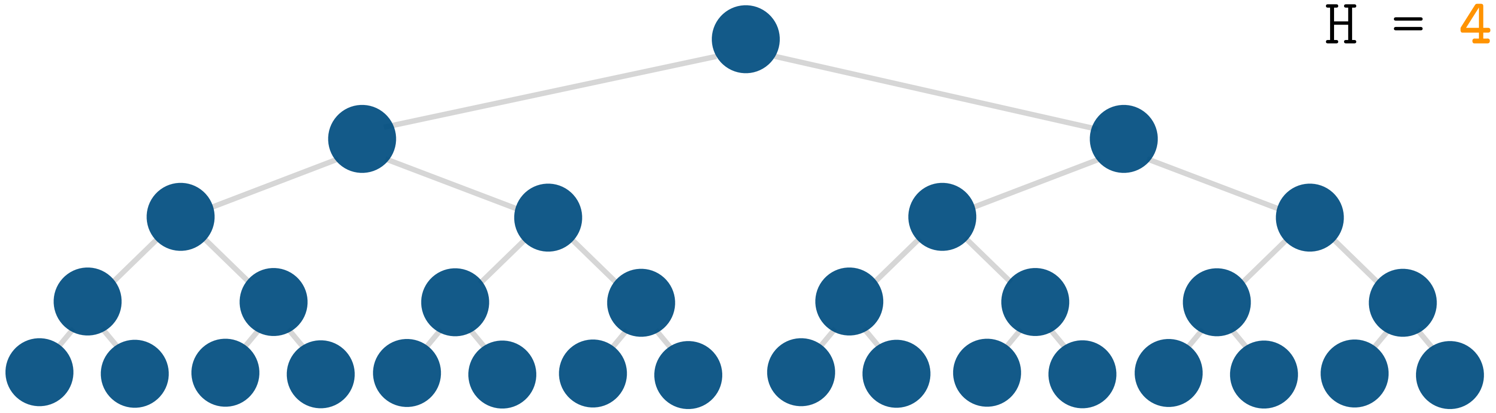
N = 3
H = 1



N = 7
H = 2



N = 15
H = 3



N = 31
H = 4

Perfect Binary Trees

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$$\# \text{ of nodes } (N) = 2^{\text{height } + 1} - 1$$

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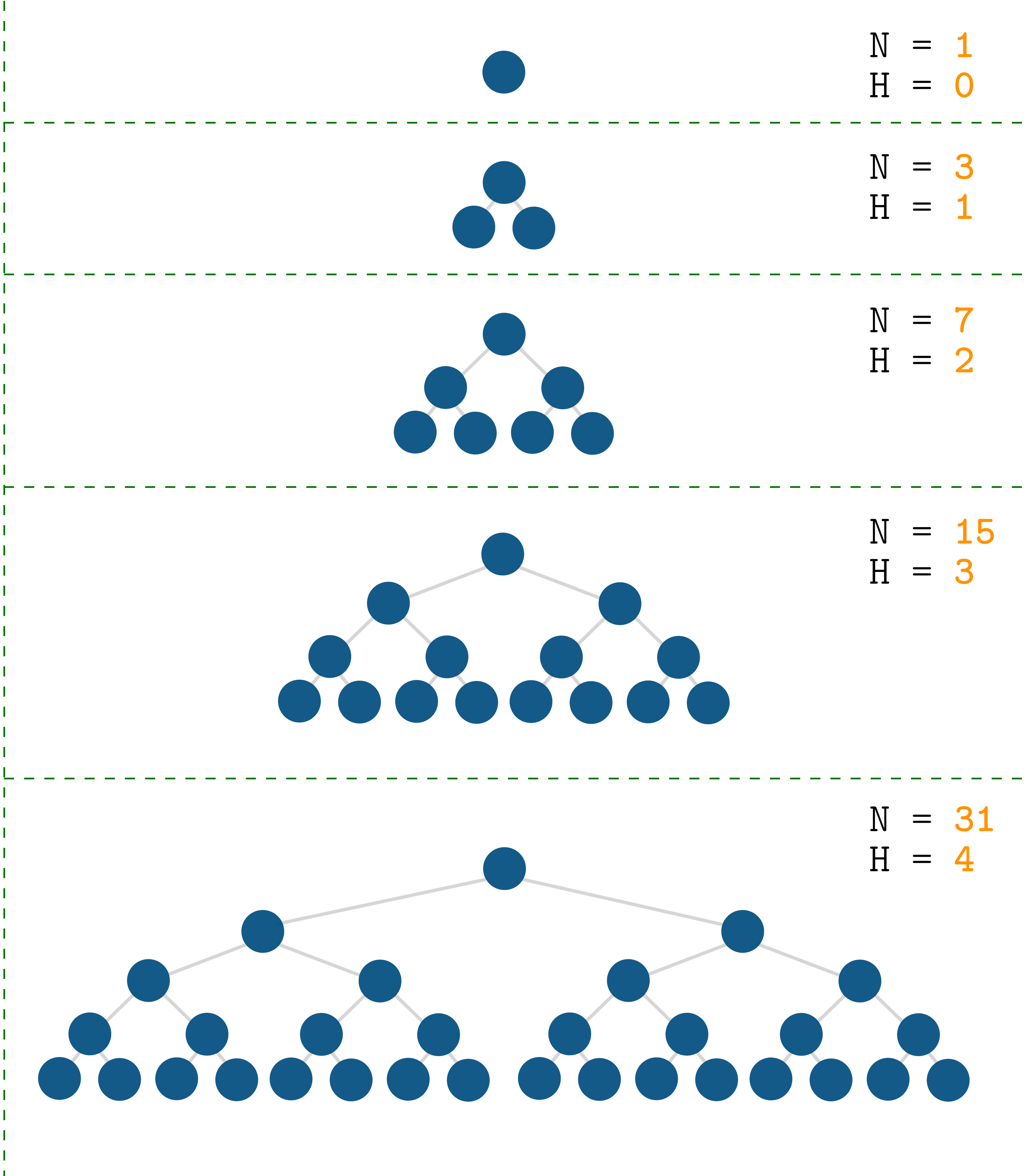

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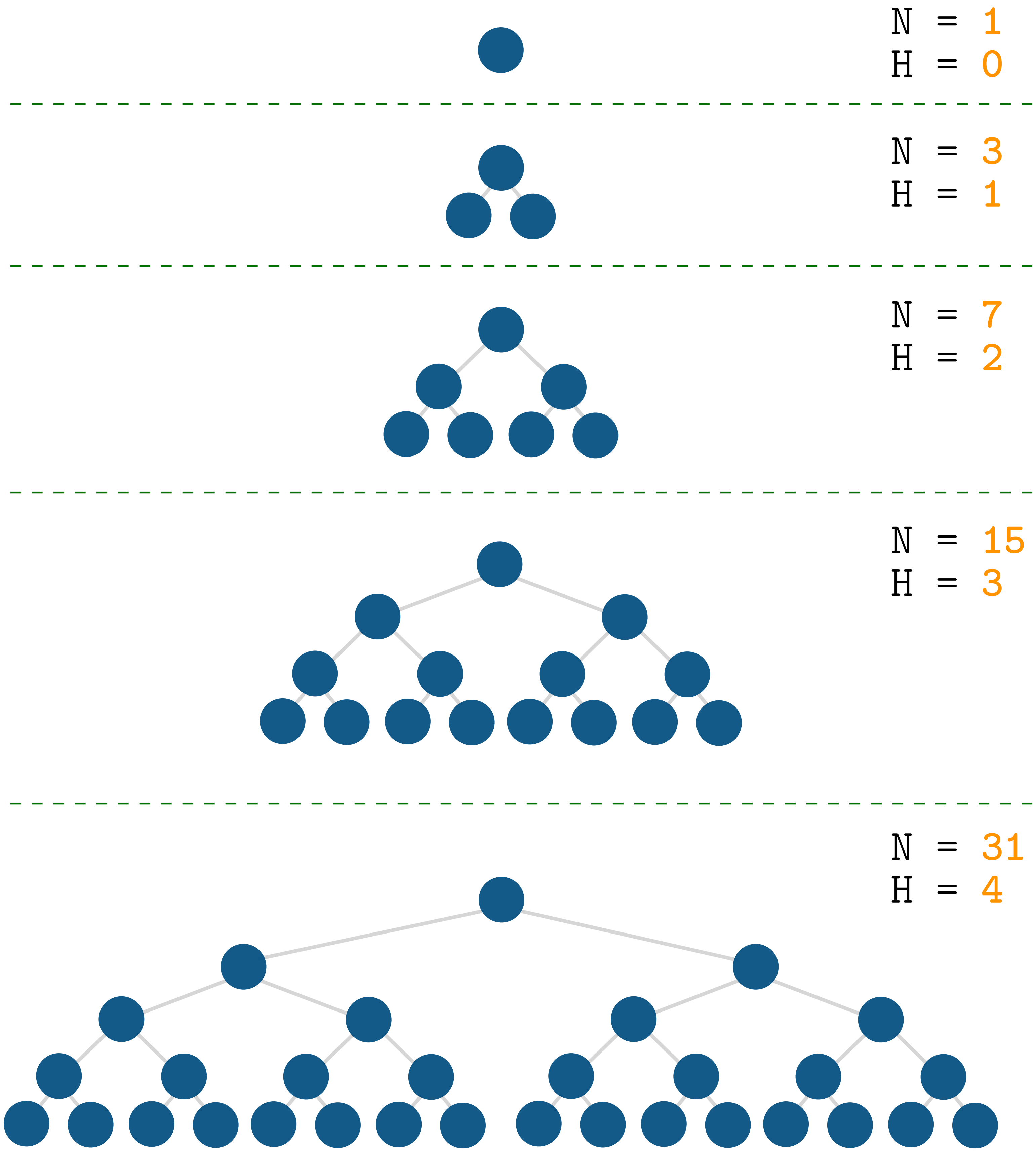
IMPORTANT!

The height of a *perfect* binary tree is *logarithmic* in the number of nodes in the tree!



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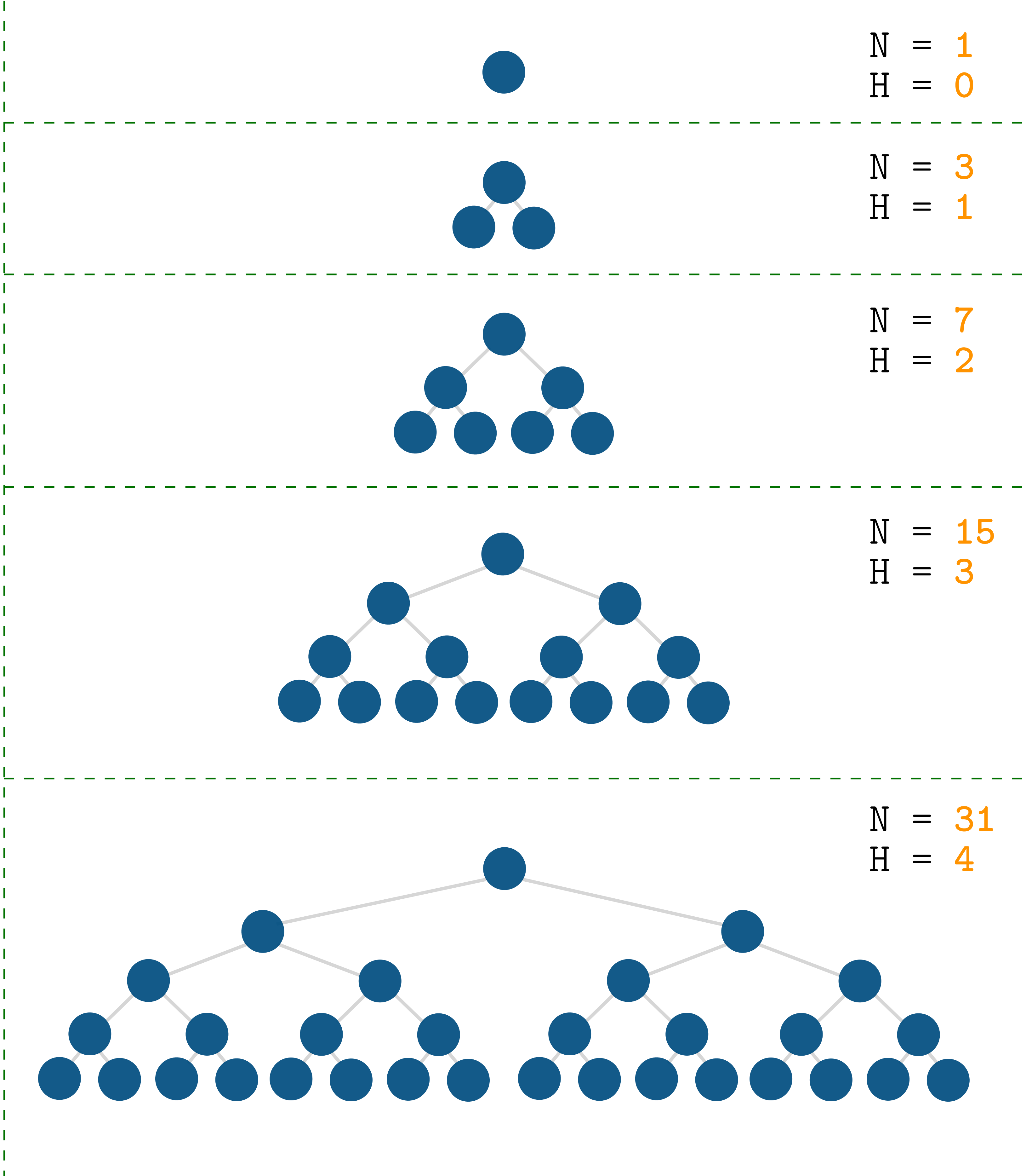
$$\text{Height} = \log_2(N + 1) - 1$$

$$\text{\# of leafs} = 2^{\text{height}}$$

$$\text{\# of leafs} = \frac{N + 1}{2}$$

$$\text{\# of internal nodes} = \frac{N - 1}{2}$$



Balanced Binary Trees

Balanced Binary Tree. the heights of the two child subtrees of any node differ by at most one.

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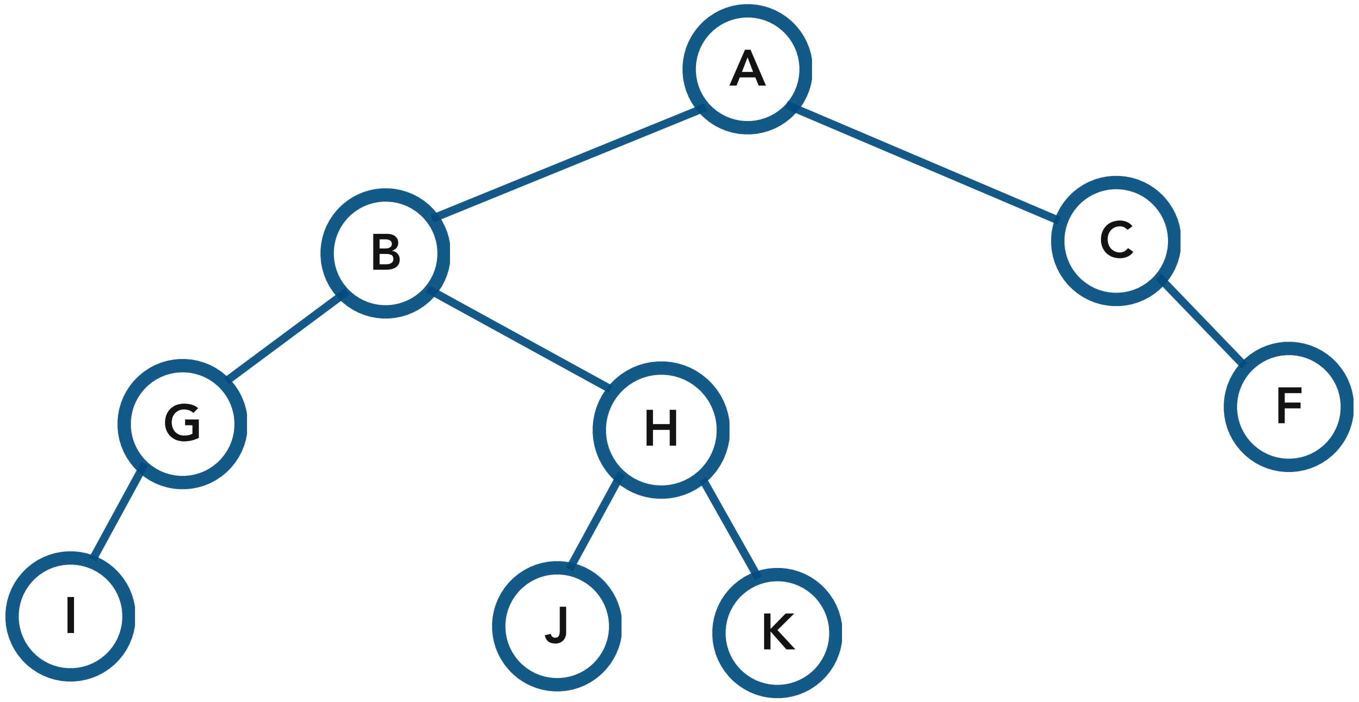
Balance Factor of a node (bf) = height of left child – height of right child



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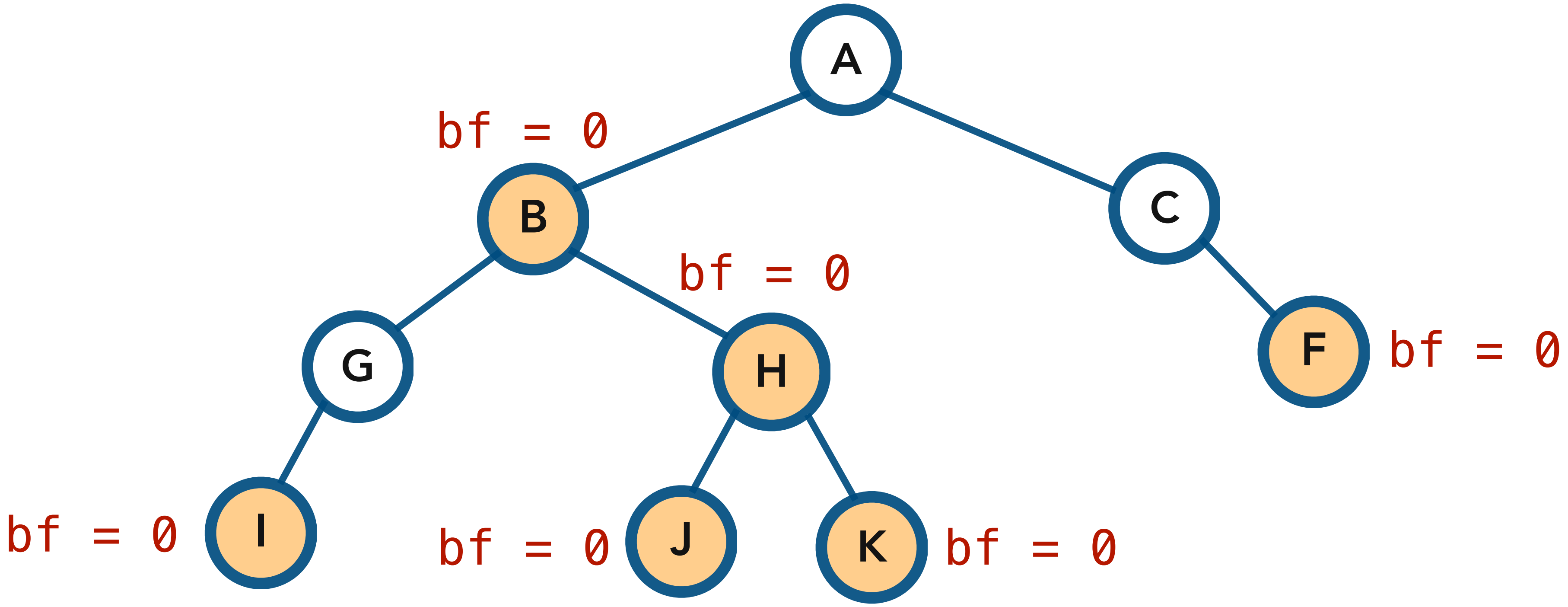
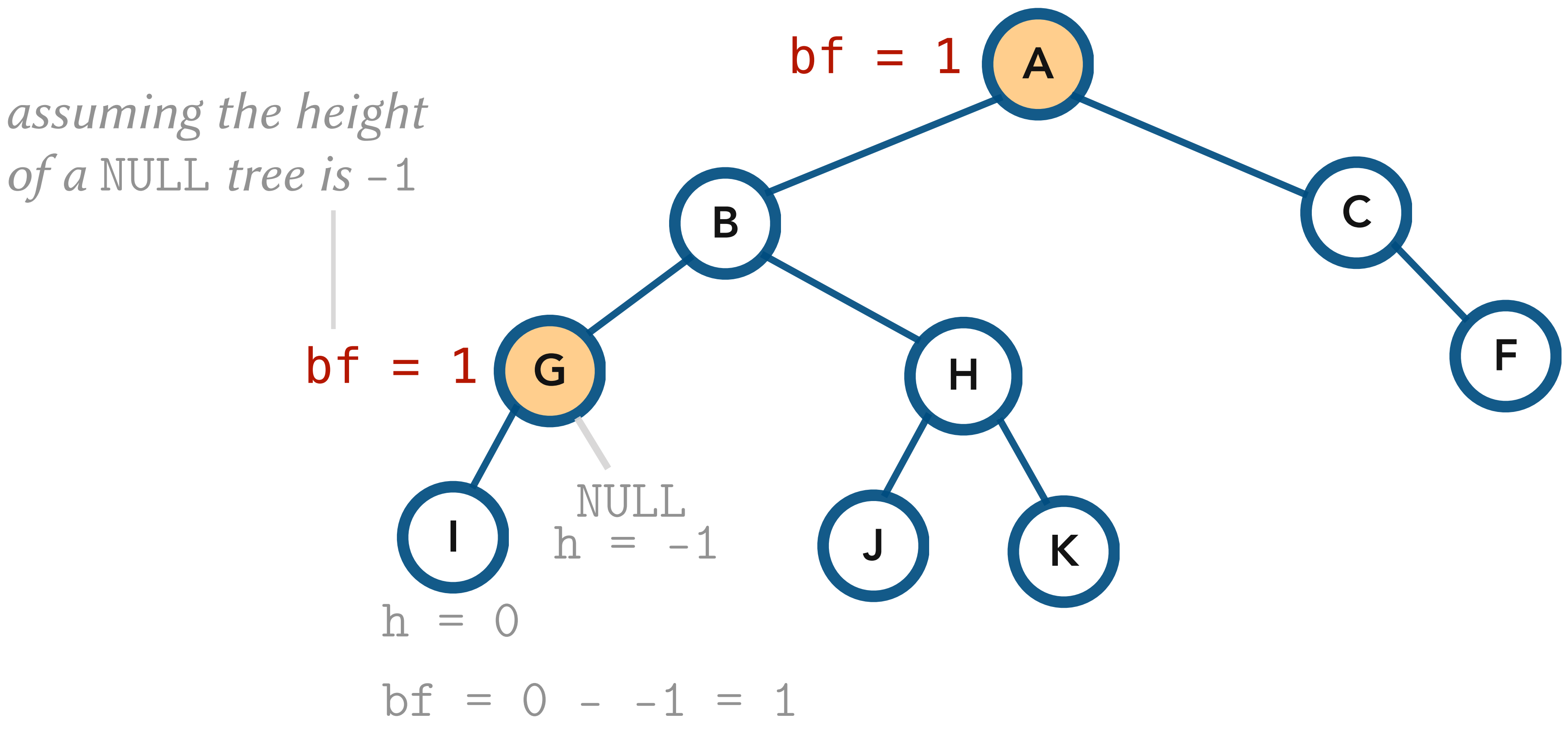


Balanced Binary Trees

Balanced Binary Tree. the heights of the two child subtrees of any node differ by at most one. I.e. the balance factor for every node = 0, 1 or -1.

Balance Factor of a node (bf) = height of left child – height of right child



assuming the height of a NULL *tree is* -1

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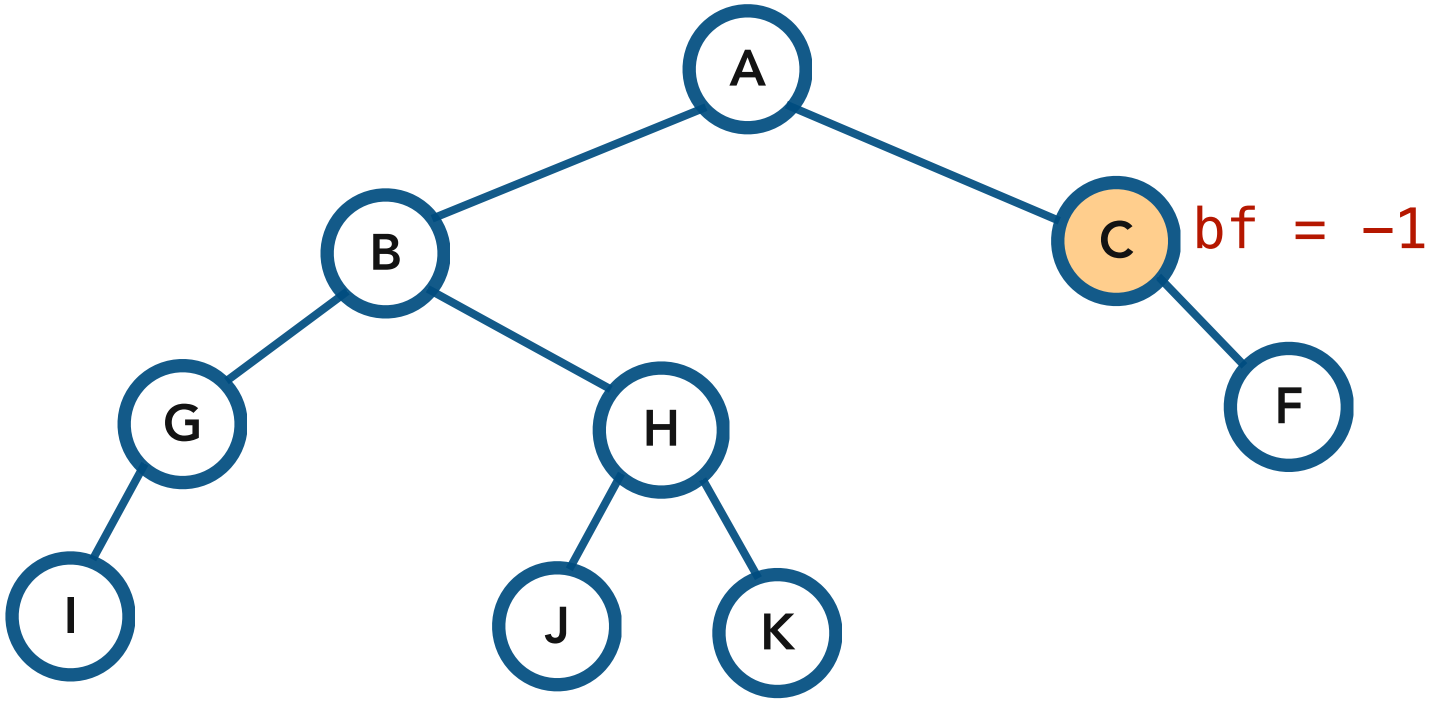

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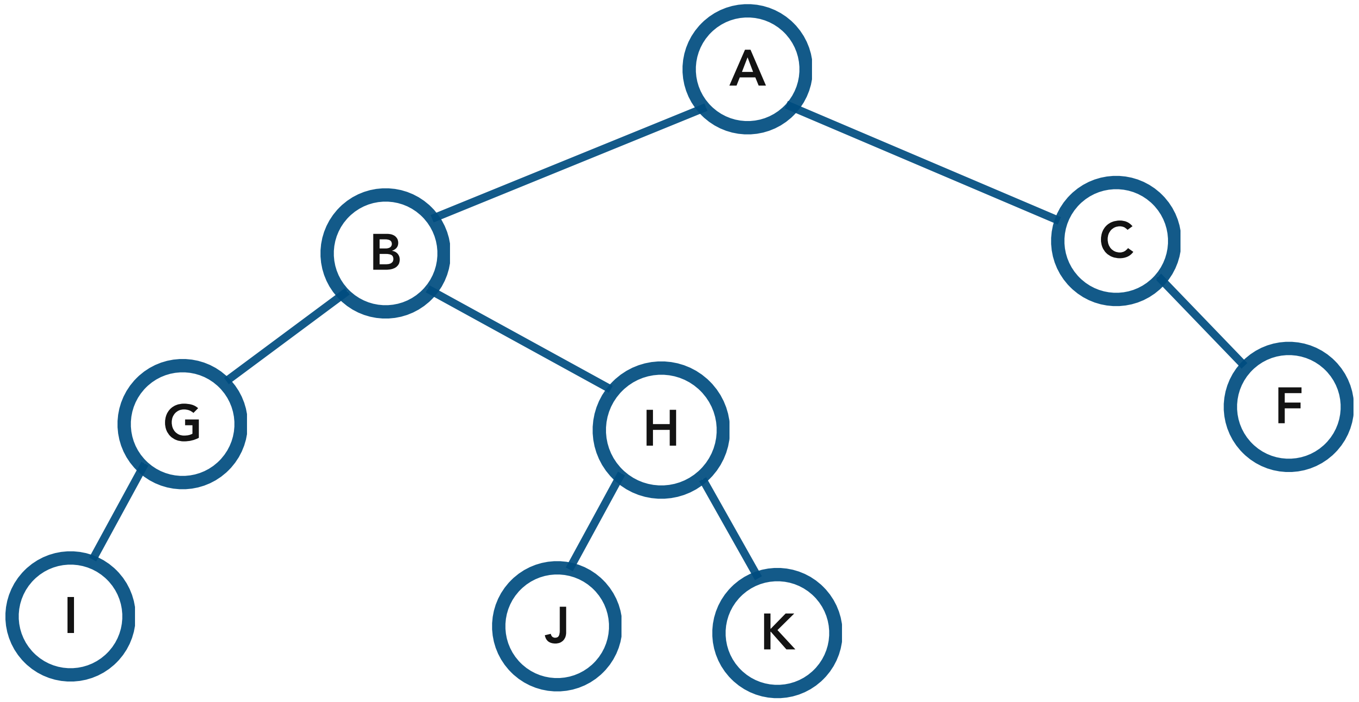
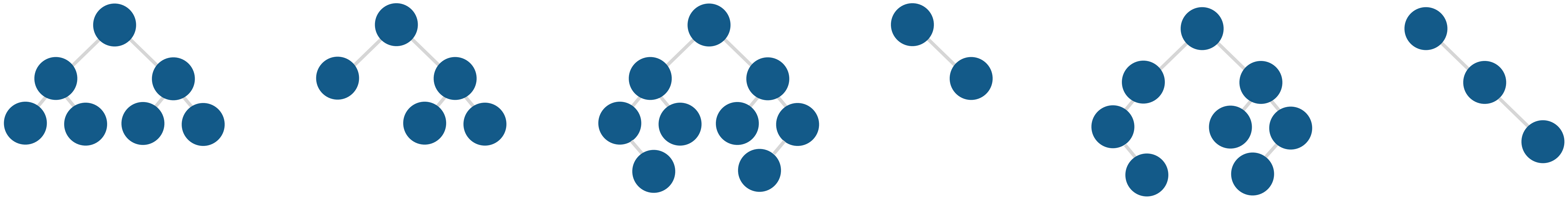


Which trees are balanced?



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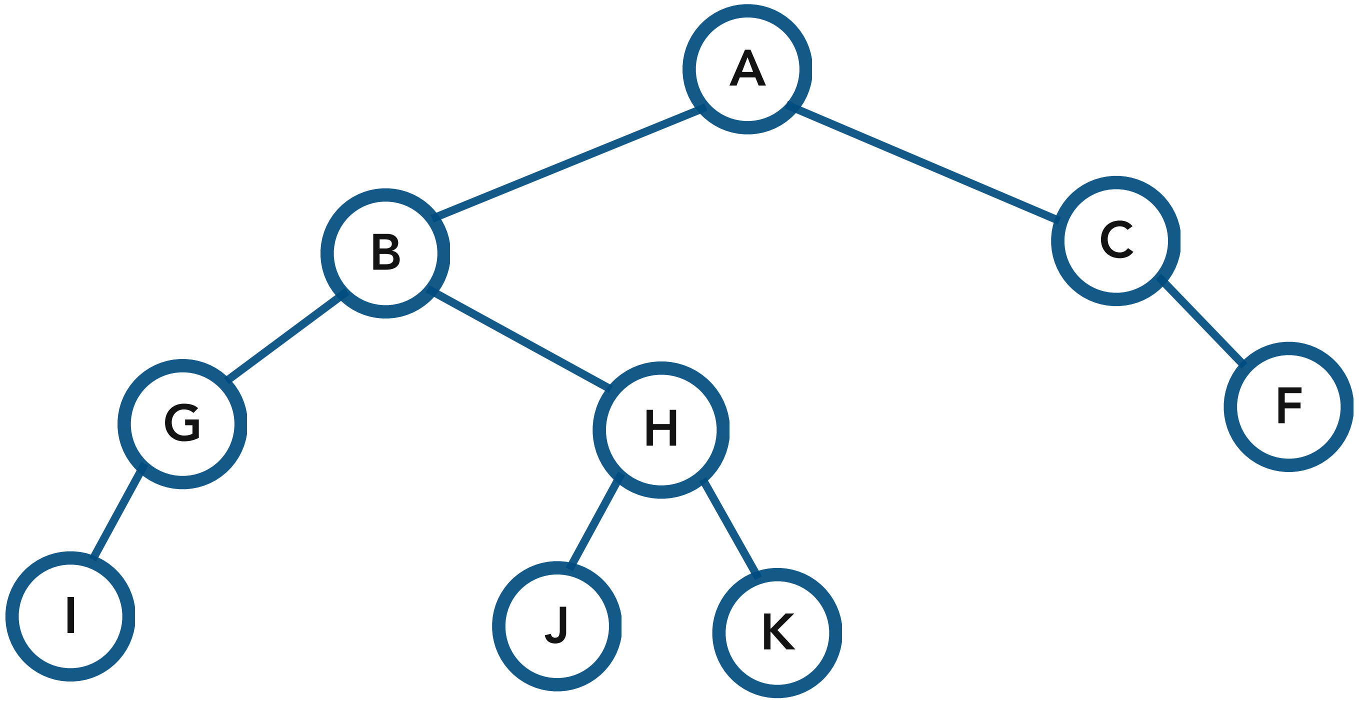

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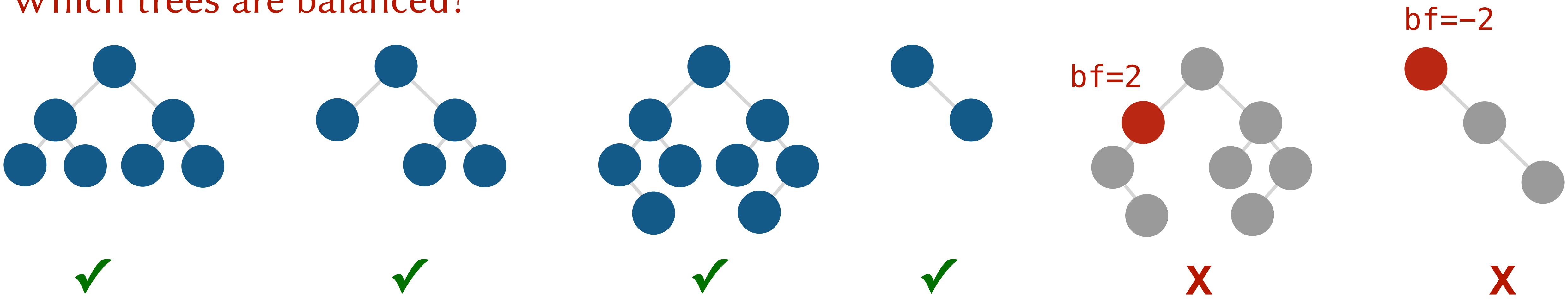

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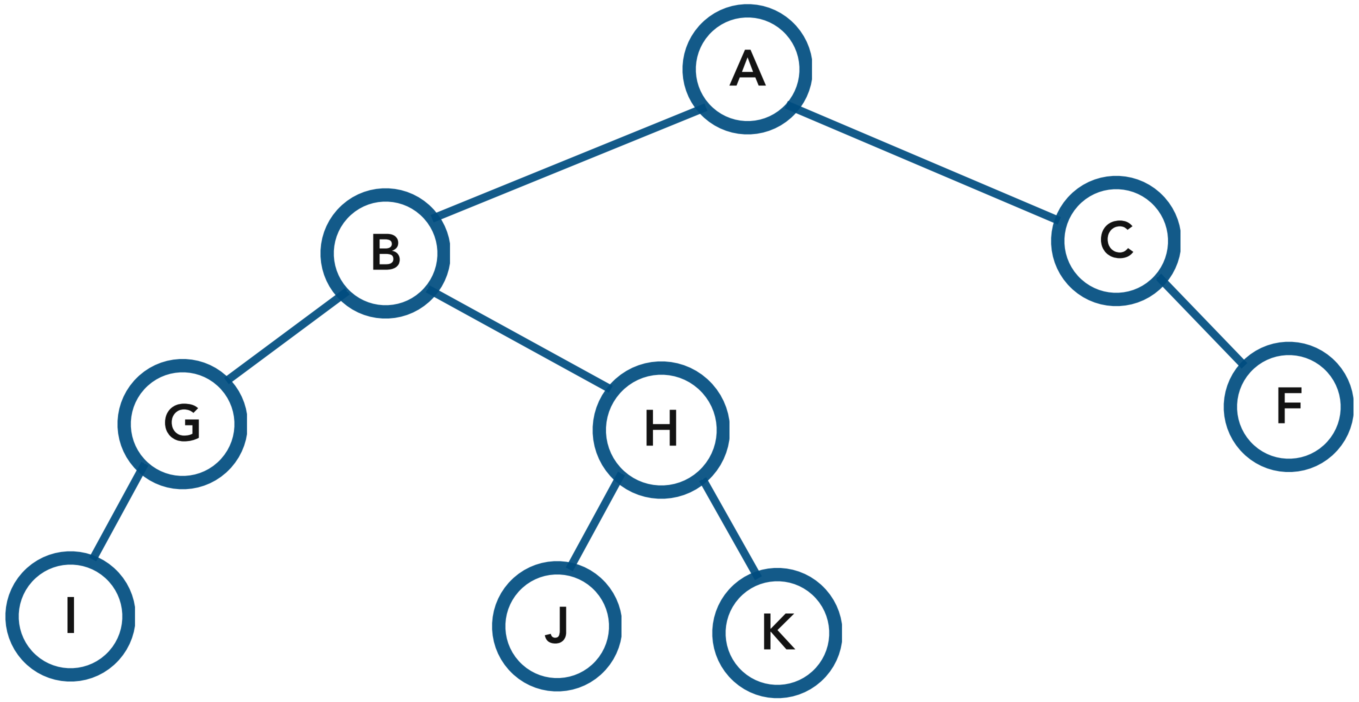
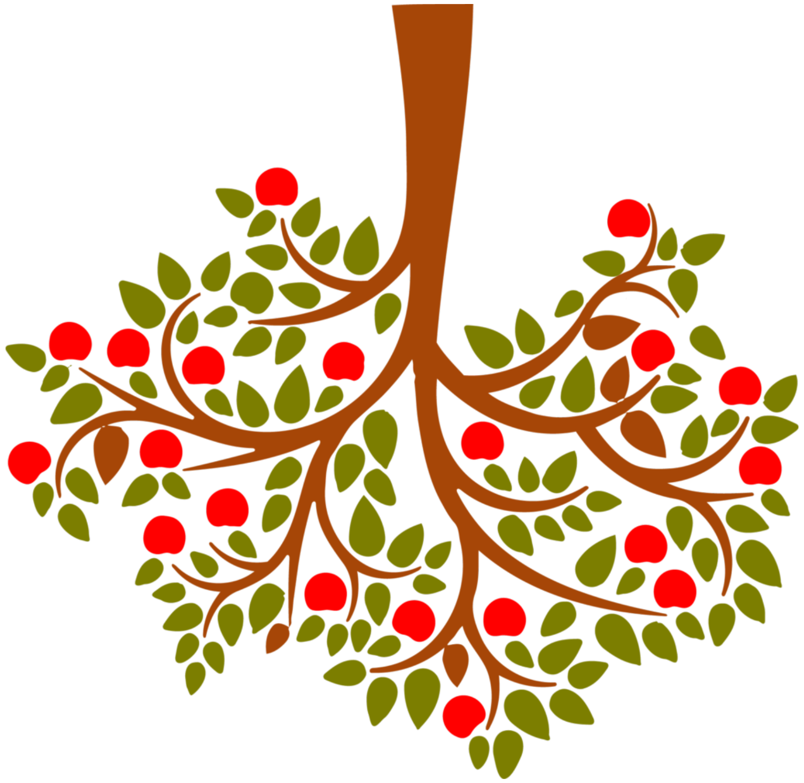


IMPORTANT!

The height of a *balanced* binary tree is *logarithmic* in the number of nodes in the tree.



Tree Data Structures

- Definitions and properties
- ● Basic Operations
- Balanced binary search trees
- Tree traversals