Asymptotic Notation Exercises

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Exercise 1

Prove each of the following statements.

1. $3n^4 + 100n^2 + 42n = O(n^4)$
2. $2^{2n} \neq O(2^n)$
3. $2^n = \Omega(2^{\frac{n}{2}})$
4. $n! = O((n+1)!)$
5. $(n+b)^2 = \Theta(n^2)$ for any $b \geq 1$.
6. $\sum_{i=1}^{n}(2i-1) = \Theta(n^2)$
7. $\sum_{i=0}^{n} 2^i = \Theta(2^n)$
8. $3n + 1 = o(n^2)$
9. $k^n = o((k+1)^n)$, where k is a constant that is > 0.
10. $3n^2 + 1 = \omega(n)$

Exercise 2

For each of the following groups of functions, rearrange the functions in the group based on their order of growth, such that if a function f appears before a function g then f must be $O(g)$.
Assume the base of the logarithm to be 2 if the base makes a difference.

1. $2^n,\ 2^{\log n},\ 2^{n^2},\ 2^{2^{\log n}},\ 2^{\frac{1}{n}}$
2. $\log n,\ \log(\log n),\ \log^2 n,\ \log^4 n,\ n,\ \log(n^{\log n})$
3. $5^{\frac{n}{2}},\ 2^n,\ 7^{\log n},\ 2^{3000},\ 1.0001^n$
4. $n,\ (\log n)^{\log n},\ (\log n)!,\ 2^n,\ \frac{1}{n},\ \frac{1}{2^n}$

Exercise 3

For each of the following pairs of functions *f* and *g*, mention which of the shown relations apply (assume *k* is a positive constant).

f	g	$f = O(g)$	$f = \Omega(g)$	$f = \Theta(g)$	$f = o(g)$	$f = \omega(g)$
kn	n					
kn	k					
$n^{\log k}$	$k^{\log n}$					
$\log^k n$	n					
k^n	k^{n+k}					
k^n	k^{kn}					
n^k	n^{k+1}					
k^k	$k+1$					
$n^{\frac{1}{k}}$	$\log_k(n)$					

Exercise 4

Provide a counterexample for each of the following statements:

1. If $f(n) = O(n)$ and $g(n) = O(n^2)$ then f grows slower than g.
2. If $f(n) = O(1)$, then $f(n) = \Theta(1)$.
3. For any two functions $f(n)$ and $g(n)$, $f(n) = O(g(n))$ or $f(n) = \Omega(g(n))$.
4. If $f(n) = O(g(n))$, then $2^f = O(2^g)$

Solutions

1.1 $3n^4 + 100n^2 + 42n = O(n^4)$

We need to show that there exist two constants *c* and n_0 such that:

$0 \le 3n^4 + 100n^2 + 42n \le cn^4$ for all $n \ge n_o$

We know that:

$0 \le 3n^4 + 100n^2 + 42n \le 3n^4 + 100n^4 + 42n^4 \le 145n^4$ for all $n \ge 1$

We can pick $c = 145$ and $n_o = 1$

1.2 $2^{2n} \ne O(2^n)$

Let's assume for the sake of contradiction that there exist two constants *c* and n_0 such that:

$0 \le 2^{2n} \le c2^n$ for all $n \ge n_o$

If we divide by 2^n the inequality becomes:

$0 \le 2^n \le c$ for all $n \ge n_o$

This is clearly false because 2^n is an increasing function that approaches ∞ as n increases, so it cannot always remain below a constant, regardless of what this constant is. Hence, the initial assumption is false.

1.3 $2^n = \Omega(2^{\frac{n}{2}})$

We need to show that there exist two constants *c* and n_0 such that:

$0 \le c \times 2^{\frac{n}{2}} \le 2^n$ for all $n \ge n_o$

If we divide by $2^{\frac{n}{2}}$ the inequality becomes:

$0 \le c \le 2^{\frac{n}{2}} \le \sqrt{2}^n$ for all $n \ge n_o$

We can pick $c = 1$ and $n_o = 1$ since $1 \le \sqrt{2}^1$ and $\sqrt{2}^n$ is an increasing function, so the inequality is always true for all $n \ge 1$.

1.4 $n! = O((n+1)!)$

We need to show that there exist two constants *c* and n_0 such that:

$0 \leq n! \leq c \times (n+1)!$ for all $n \geq n_o$

If we divide by $n!$ the inequality becomes:

$0 \leq 1 \leq c \times (n+1)$ because $(n+1)! = (n+1) \times n!$

Hence, we can pick $c = 1$ and $n_o = 1$

1.5 $(n+b)^2 = \Theta(n^2)$ for any $b \geq 1$.

1. We need to show that there exist two constants *c* and n_0 such that:

$0 \leq (n+b)^2 \leq cn^2$ for all $n \geq n_o$ and for any $b \geq 1$.

Since $(n+b)^2 = n^2 + 2bn + b^2$:

$n^2 + 2bn + b^2 \leq n^2 + 2bn^2 + b^2n^2 \leq (1 + 2b + b^2)n^2$ for all $n \geq 1$

We can pick $c = 1 + 2b + b^2$ and $n_o = 1$
This means that $(n+b)^2 = O(n^2)$ for any $b \geq 1$.

2. We need to show that there exist two constants *c* and n_0 such that:

$0 \leq cn^2 \leq (n+b)^2$ for all $n \geq n_o$ and for any $b \geq 1$.

We can pick $c = 1$ and $n_o = 1$

$0 \leq n^2 \leq n^2 + 2bn + b^2$
This is true because $n \geq 1$ and $b \geq 1$

1.6 $\sum_{i=1}^{n}(2i-1) = \Theta(n^2)$

The left hand side is equivalent to:
$2 \times \sum_{i=1}^{n} i - \sum_{i=1}^{n} 1 = n(n+1) - n = n^2 + n - n = n^2$
Therefore, the question is to show that $n^2 = \Theta(n^2)$ Which is true given the properties $f = O(f)$ and $f = \Omega(f)$, which imply that $f = \Theta(f)$

1.7 $\sum_{i=0}^{n} 2^i = \Theta(2^n)$

The left hand side is equivalent to $2^{n+1} - 1$. Hence, we need to show that:

1. There exist two constants *c* and *n_0* such that $0 \leq 2^{n+1} - 1 \leq c \times 2^n$ for all $n \geq n_o$

If we pick $c = 2$, the inequality becomes:

$0 \leq 2^{n+1} - 1 \leq 2^1 \times 2^n \leq 2^{n+1}$, which is clearly true for all $n \geq 1$.
This means that $2^{n+1} - 1 = O(2^n)$

2. There exist two constants *c* and *n_0* such that $0 \leq c \times 2^n \leq 2^{n+1} - 1$ for all $n \geq n_o$

If we pick $c = 1$ and $n_o = 1$, the inequality becomes:

$0 \leq 2^n \leq 2^{n+1} - 1 \leq 2 \times 2^n - 1 \leq 2^n + (2^n - 1)$, which is true for all $n \geq 1$ because $2^n - 1 \geq 1$ for all $n \geq 1$.

This means that $2^{n+1} - 1 = \Omega(2^n)$

1.8 $3n + 1 = o(n^2)$

We will prove the equivalent statement $3n + 1 \neq \Omega(n^2)$.

Let's assume for the sake of contradiction that $3n + 1 = \Omega(n^2)$. I.e., there exist two constants *c* > 0 and *n_0 > 0* such that: $0 \leq cn^2 \leq 3n + 1$ for all $n \geq n_o$

Dividing the inequality by n gives: $0 \leq cn \leq 3 + \frac{1}{n}$ for all $n \geq 1$

This statement is clearly false because:

1. cn is an increasing function that approaches ∞ regardless of the value of c.
2. The values of $3 + \frac{1}{n}$ are between 4 and 3.

1.9 k^n = $o((k+1)^n)$, where k is a constant that is > 0.

We will prove the equivalent statement $k^n \neq \Omega((k+1)^n)$.

Let's assume for the sake of contradiction that $k^n = \Omega((k+1)^n)$. I.e., there exist two constants *c* > 0 and *n_0 > 0* such that: $0 \leq c(k+1)^n \leq k^n$ for all $n \geq n_o$

This inequality is false if $\log_k(c(k+1)^n) \leq \log_k(k^n)$ is false.

$$\log_k(c(k+1)^n) \le \log_k(k^n) \longrightarrow \quad n(\log_k c + \log_k(k+1) \le n \log_k k$$
$$n \log_k c + n \log_k(k+1) \le n$$

This is clearly false because $n \log_k(k+1) > n \log_k k > n$, which makes the left hand side $> n$ regardless of what the value of c is.

1.10 $3n^2 + 1 = \omega(n)$

We will prove the equivalent statement $3n^2 + 1 \ne O(n)$.

Let's assume for the sake of contradiction that $3n^2 + 1 = O(n)$. I.e., there exist two constants *c* > 0 and *n₀* > 0 such that: $0 \le 3n^2 + 1 \le cn$ for all $n \ge n_o$

Dividing the inequality by n gives: $0 \le 3n + \frac{1}{n} \le c$ for all $n \ge 1$

This statement is false because $3n + \frac{1}{n}$ is an increasing function that approaches ∞ as n increases and c is a constant.

2.1 $2^{\frac{1}{n}},\ 2^{\log n},\ 2^n,\ 2^{2^{\log n}},\ 2^{n^2}$

Notes.

- The values for $2^{\frac{1}{n}}$ decrease when n increases (from 2 when $n = 1$ to 1 when $n = \infty$). This makes the function $\Theta(1)$ (i.e. there is a constant that is always $\ge 2^{\frac{1}{n}}$ and there is a constant that is always $\le 2^{\frac{1}{n}}$).
- $2^{\log_2 n} = n^{\log_2 2} = n$
- $2^{n^2} = (2^n)^n$
- $2^{2^{\log 2}} = 2^n$

2.2 $\log(\log n),\ \log n,\ \log(n^{\log n}),\ \log^2 n,\ \log^4 n,\ n$

Notes.

- $\log(n^{\log n}) = \log^2 n$
- One way to reason about this is to give the function $\log n$ the name x (for example). This makes the functions as follows:
  $\log x,\ x,\ \log(n^x) = x^2,\ x^2,\ x^4,\ 2^x$

2.3 $2^{3000},\ 7^{\log n},\ 1.0001^n,\ 2^n,\ 5^{\frac{n}{2}}$

Notes.

- $7^{\log_2 n} = n^{\log_2 7} = n^{2.81}$
- $5^{\frac{n}{2}} = \sqrt{5}^n \approx 2.236^n$

2.4 $\frac{1}{2^n},\ \frac{1}{n},\ n,\ (\log n)!,\ (\log n)^{\log n},\ 2^n$

Notes.

- The values for $\frac{1}{n}$ decrease from 1 to 0 as the values of n increase. Therefore $\frac{1}{n} = \Theta(1)$ (i.e. There is a constant that is always $\geq \frac{1}{n}$ and there is also a constant that is always $\leq \frac{1}{n}$).
- The values for $\frac{1}{2^n}$ decrease from 1 to 0 as the values of n increase. Therefore $\frac{1}{2^n} = \Theta(1)$.
- One way to reason about this is to give the function $\log n$ the name x (for example). This makes the functions as follows:

$n = 2^{\log n} = 2^x$
$(\log n)! = x!$
$(\log n)^{\log n} = x^x = (2^{\log x})^x = 2^{x \log x}$
$2^n = 2^{2^{\log n}} = 2^{2^x}$

3

f	g	$f = O(g)$	$f = \Omega(g)$	$f = \Theta(g)$	$f = o(g)$	$f = \omega(g)$
kn	n	✓	✓	✓		
kn	k		✓			✓
$n^{\log k}$	$k^{\log n}$	✓	✓	✓		
$\log^k n$	n	✓			✓	
k^n	k^{n+k}	✓	✓	✓		
k^n	k^{kn}	✓			✓	
n^k	n^{k+1}	✓			✓	
k^k	$k+1$	✓	✓	✓		
$n^{\frac{1}{k}}$	$\log_k(n)$		✓			✓

4.1 If $f(n) = O(n)$ and $g(n) = O(n^2)$ then f grows slower than g.

Let $f(n) = n$ and $g(n) = \log(n)$, then:

$n = O(n)$ is true and $\log(n) = O(n^2)$ but n does not grow slower than $\log(n)$.

4.2 If $f(n) = O(1)$, then $f(n) = \Theta(1)$.

Let $f(n) = \frac{1}{n}$.

- Since $0 \leq f(n) \leq 1$, We can multiply 1 by a constant c such that $0 \leq f(n) \leq c \times 1$ for all $n \geq 0$. Hence, $f(n) = O(1)$.
- Since $f(n)$ approaches 0 when n approaches ∞, no constant $c > 0$ can be multiplied by 1 such that $0 \leq c \times 1 \leq f(n)$ for all $n \geq$ some n_o. Hence, $f(n) \neq \Omega(1)$.

4.3 For any two functions $f(n)$ and $g(n)$, $f(n) = O(g(n))$ or $f(n) = \Omega(g(n))$.

Let $f(n) = cos(n)$ and $g(n) = sin(x)$.

Multiplying any of the two functions by a constant does not make it always greater or always less than the other function.

4.4 If $f(n) = O(g(n))$, then $2^f = O(2^g)$

Let $f(n) = \log_2 n$ and $g(n) = \frac{1}{2} \log_2 n$.

It is clear that $\log_2 n = \frac{1}{2} \log_2 n$ (pick $c = 2$ and $n_o = 1$). However, $2^{\log_2 n} = n \neq O(2^{\frac{1}{2} \log_2 n})$, because $2^{\frac{1}{2} \log_2 n} = \sqrt{n}$.